Large random matching markets with localized preference structures can exhibit large cores *

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Abstract

I present models of matching markets with non-homogeneous agent preferences (drawn from the computer science literature on network structure). Some plausible preference structures support significant incentives to manipulate matching outcomes. Established theoretical approaches and simulation techniques show that the scope for manipulation remains substantial as such markets become large and unbalanced, contrasting prior work on homogeneous preferences, which finds little such scope. Scope for manipulation corresponds to core size and differences in agents' welfare between core outcomes. These results suggest that largeness and cross-side imbalance are insufficient to explain empirical observations of small cores in matching markets; I discuss alternative explanations.

1 Introduction

I consider matching markets where both sides have preferences over potential matches, and no common numeraire (or contract details) allow for transfer of utility. Examples of importance and interest include allocation

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‡This text presents independent work and only the opinions of the named author.

of students to schools at primary through graduate levels, allocation of migrants to settlement destinations, matching of participants in bipartite dating pools, construction of exclusive social organizations, and some settings of assignment of entry-level workers to employer firms.

As matching markets are used in practice to coordinate and allocate the labor, training, and education of hundreds of thousands of participants each year—typically in multi-year commitments—their efficiency can be of substantial economic importance. Correspondingly, the outcomes of matching can be of such importance to participants that markets provide significant incentives for strategic manipulation by sophisticated participants (Pathak and Sönmez (2008)). Thus, matching-market designers are directly concerned with the choices and incentives that mechanisms offer to market participants, and their effects on allocation dynamics in equilibrium (Roth (2002)).

Previous theoretical studies of matching markets have suggested that agents' abilities and incentives to manipulate generally vanish as the size of the market grows (*e. g.*, Immorlica and Mahdian (2005)). I find, however, that these theoretical results depend crucially on the preferences of agents lacking locality structure; in matching markets with localized preference structures, agents' incentives to manipulate can instead robustly *fail* to vanish. This suggests that observations of large markets with negligible scope for manipulation (and localized preferences) may not be explained by existing theory of matching-market structure. Alternate explanations are required, and will require further structural analysis of matching markets of interest.

1.1 Motivation and background

The literature on matching-market mechanism design has shown that *ex post stability* of outcomes, and *ex ante incentive-compatibility* of mechanisms, are of primary importance to the success of an allocation marketplace (Roth (1984, 2002)). Intuitively speaking, *ex post* stability is the condition that it is an equilibrium for agents to follow the assigned allocation (rather than seeking a match outside it, or electing to later recontract). Gale and Shapley (1962) have shown that, in one-to-one[1] matching markets, stable allocations exist in general and are efficiently computable. In fact, stable-allocation-finding mechanisms are now used in a variety of settings where

[1] In many-to-one and many-to-many settings, existence of stable allocations generally requires either full substitutability (Roth and Sotomayor (1990)) or full complementarity (Rostek and Yoder (2019)).

'unraveling' deviation from mechanism assignments would otherwise pose significant welfare costs to the market (Roth (1984); Roth and Xing (1994); Niederle and Roth (2003); Fréchette *et al.* (2007)).

However, stability does not necessarily determine a unique allocation. Multiple stable allocations may exist, and individual agents may have sufficient market power to influence which equilibrium obtains. More explicitly, a nontrivial *core* (which, in this setting, coincides with the set of stable allocations) excludes the possibility of a core-selecting mechanism for which truth-telling is a dominant strategy for all participants. An agent who selects a desirable allocation from the true core—and (mis)represents that only that allocation's outcome is acceptable to them—will be granted that outcome by a core-selecting mechanism (Gale and Sotomayor (1985)). Thus, agents deviating from truth-telling can manipulate a core-selecting mechanism into assigning them their most-favored outcome among those achieved in the core.

Therefore, *ex ante* incentive-compatibility[2] of a core-selecting mechanism can only be assured for agents who are assigned their most-favored core outcome when truth-telling. Knuth (1976) demonstrates that this can, at best, be simultaneously assured for all agents on one side of the market (unless only one core allocation exists).

It follows that the available incentives for strategic manipulation are determined by the gap in welfare (of the mechanism-disfavored side) between the subjectively optimal and pessimal core outcomes.[3] Insofar as strategic manipulation degrades welfare by risking allocative inefficiency (and creating its own wasteful investment race), it threatens the primary objectives of mechanism design. Agents' welfare gaps among core allocations are therefore of direct concern to the matching-market designer.

Prior work has investigated within-core welfare gaps by bounding core size. Roth and Peranson (1999) observed qualitatively small cores in the National Resident Matching Program and they conjectured, from simulation results, that (uniformly-)randomly-drawn fixed-size preference lists yielded vanishingly small cores as markets became large. Immorlica and Mahdian (2005) proved this conjecture. Kojima and Pathak (2009) and Ashlagi *et al.* (2017) demonstrated comparable results for balanced and unbalanced many-to-one markets, respectively. I further discuss prior work on core size in large

[2] *i. e.*, compatibility of a truth-telling strategy with agents' own incentives.

[3] Furthermore, the presence of manipulation by the mechanism-disfavored side of the market creates game-theoretic incentives for the mechanism-*favored* side of the market to manipulate in order to discourage the other side's manipulation. Gale and Sotomayor (1985) analyze the strategic equilibrium of this game in the perfect-information setting.

matching markets in § 5.1 below.

Thus far, the literature has considered classes of *homogeneous* models of markets under which core size (and welfare gap size) vanishes as the number of agents in the market becomes large, and as agents' preferences over partners become more correlated. The present work contrasts these results by presenting a class of *non-homogeneous* market models. These models are inspired by analogy to a class of models for inter-agent networks drawn from the computer science literature on network structure, which were originally presented to model empirical observations of network dynamics.[4] I demonstrate that, under non-homogeneous models where agents' preferences exhibit sufficient locality, optimal–pessimal welfare gaps remain substantial as markets become large. Under some specifications, welfare gaps *grow* as local preference correlation increases.

1.2 Overview of results

My model of college admissions markets extends the Kojima and Pathak (2009) model of large matching markets. Their model is characterized by (a) a large number of colleges, of which each student finds only a small number acceptable, and (b) random, conditionally independent agent preferences. I extend this model by weakening the latter condition and allowing a student's opinion of one college to correlate with their opinion of another (though I maintain *ex ante* symmetry between schools, and between students, for my main results).

By introducing locality to the structure of student preferences, I find that the core is non-vanishing under certain simple and natural preference structures. Furthermore, introducing preference locality to the Ashlagi *et al.* (2017) model of *unbalanced* random matching markets, I find that even markets with significant imbalance (in the number of agents on each side of the market) can have large cores. While these results do not contradict the theoretical conclusions of the prior literature on random large markets, they suggest that mere market largeness may not be sufficient to explain observations of small cores when preference locality is present.

I augment these theoretical results with simulation results that demonstrate that core size and manipulation incentives are substantial in magnitude in practice. These simulations demonstrate the persistence of a large core under significant imbalance, and show that the otherwise sharp advantage afforded to the short side of a homogeneous market is attenuated by

[4] I further discuss the relevant network-structure literature in § 5.2 below; for a more thorough treatment, see Chapter 2 of Rheingans-Yoo (2016).

preference locality. Finally, I discuss plausible alternative explanations for empirical observations of small cores in some large matching markets with preference locality.

2 Model

I extend the random matching market model of Kojima and Pathak (2009) to support non-homogeneous structure among students' preferences. This model likewise embeds the "correlated preferences" models of Ashlagi *et al.* (2017), allowing for direct comparison of homogeneous and non-homogeneous preference structures.

2.1 Matching markets, core size, and market power

I consider one-to-one and many-to-one matching markets, with *colleges* demanding $q \geq 1$ matches in general and *students* demanding precisely one match. Appendix A presents standard formal definitions of matching markets and stability.

In matching markets, the *core*[5] coincides with the set of stable matchings (Roth (1985)). Following Immorlica and Mahdian (2005), I consider *core size* in terms of the fraction of agents with multiple stable matches[6] and formally describe the core as *small* or *vanishing* if the expected fraction of agents with multiple stable matches asymptotically vanishes as a market becomes large in the number of agents. Correspondingly, the core is formally *large* or *non-vanishing* if the expected fraction fails to vanish as the market becomes large.

In matching markets, core size creates scope for agents to manipulate core-selecting mechanisms (Gale and Sotomayor (1985)). Kojima and Pathak (2009) argue that manipulability is best understood in terms of *market power*—the ability for an agent's strategic rejection of a proposal to affect the set of other proposals that agent will later observe—and I adopt this term in the same sense where appropriate.

2.2 Random markets

Given a set of colleges C and a set of students S, a *random market* is a tuple $\tilde{\Gamma} = \left(C, S, \mathcal{P}_C, \mathcal{P}_S\right)$, where $\mathcal{P}_C$ is a probability distribution on orderings of $S \cup \{\varnothing\}$ and $\mathcal{P}_S$ is a probability distribution on orderings of $C \cup \{\varnothing\}$. Each

[5] *i. e.*, the set of outcomes on which no coalition can unilaterally improve.

[6] *i. e.*, matches achieved in some stable matching.

random market induces a market by randomly generating preferences of each college c by drawing from $\mathcal{P}_C$ and preferences of each student s by drawing from $\mathcal{P}_S$.[7]

2.2.1 Kojima–Pathak random markets

Kojima and Pathak (2009) present a special case of this random market model for $\mathcal{P}_C$ a fixed realization of colleges' preferences and $\mathcal{P}_S$ given as follows:

- Fix $k > 0$ a positive integer.
- Fix $\mathcal{D} = (p_c)_{c \in C}$ a probability distribution on C.
- Assign each student's preferences by drawing k colleges from $\mathcal{D}$ without replacement, then appending $\varnothing$ (whereafter the order of successive colleges is immaterial).

Effectively, each students' preference-ordering of colleges is composed of k independent draws from a common distribution on colleges $\mathcal{D}$. I hereafter call random markets of this form *Kojima–Pathak random markets.*

2.2.2 A simple example of preference locality

Consider a simple linear model of preference locality structure, analogous to the one-dimensional regular ring lattice that Watts and Strogatz (1992) introduce to model locality in network structure. For a positive integer k, let a *uniform-all-students, uniform 1-dimensional k-nearest-colleges random market* (hereafter *k-nearest-colleges random market* when not otherwise qualified) be a random market $\tilde{\Gamma} = \big(C, S, \mathcal{P}_C, \mathcal{P}_S\big)$ with $\mathcal{P}_C$ the uniform distribution over (complete) permutations of students and $\mathcal{P}_S$ given as follows:

- Arrange the colleges C uniformly on a circle.

[7] The present work models random markets with an expectation over colleges' preferences $\mathcal{P}_C$, which has not been necessary in some prior work. It is necessary here because certain realizations of colleges' preferences (*e. g.*, identical preferences, or subsequences of a common linear preference ranking) force the core to collapse irrespective of students' preferences. Thus, non-vanishing-market-power results cannot be demonstrated for fixed colleges' preferences *in general*; we introduce an expectation in order to evaluate the expected frequency of large cores.

As in the referenced works, I consider the possibility of manipulations under complete information; randomness over preferences is introduced only to assess the frequency of situations in which agents have incentives to manipulate.

- Place each student s uniformly at random at a point on the circle.
- Let each student have preferences drawn uniformly at random from the k nearest colleges, with all other colleges unacceptable.

Formally speaking, identify each college $c \in C$ with a unique integral point $y_c \in \mathbb{R}/|C|\mathbb{Z}$; to generate the preferences of student s, select a point $x_s \in \mathbb{R}/|C|\mathbb{Z}$ uniformly at random, then return as s's preferences a permutation chosen uniformly at random of the colleges lying in the interval $[x_s, x_s + k)$.

Remark 1. *Immorlica and Mahdian (2015), in their Remark 2.7, consider a related preference structure, where colleges are partitioned into pairs with opposing rankings of students.*[8] *Under this* paired-colleges *model, the authors demonstrate that the fraction of agents with more than one stable match fails to vanish as the number of agents grows large.*

While the paired-colleges model serves as a counterexample to a general vanishing-core conjecture, it is an unnatural model of preference structure, and implausible in practice. The k-nearest-colleges model of the present work relaxes the synthetic construction:

- *Colleges' preferences are drawn from a common distribution, rather than constructed in opposing pairs.*
- *Students choose overlapping clusters of colleges, rather than disjoint partitions.*

In Appendix D, I further generalize the relevant conditions on preference locality and provide more general models from the network-structure literature with preference-structure analogues that support large cores.

2.3 Regular markets

Denote a *sequence of random markets* $\left(\tilde{\Gamma}^{(1)}, \tilde{\Gamma}^{(2)}, \dots\right)$, where each element $\tilde{\Gamma}^{(n)} = \left(C^{(n)}, S^{(n)}, \mathcal{P}_C^{(n)}, \mathcal{P}_S^{(n)}\right)$ is a random market in which $\left|C^{(n)}\right| = n$ is the number of colleges. A sequence of random markets is $(k, \overline{q})$-*regular* if there exist positive integers k and $\overline{q}$ such that:

- For all n, and all preference-orderings $\tilde{\succ}$ supported in $\mathcal{P}_S^{(n)}$, exactly k colleges are acceptable under $\tilde{\succ}$.
- Letting q_c be the quota of students that college c can accept, $q_c \leq \overline{q}$ for $c \in C^{(n)}$ for all n.

[8] I thank an anonymous referee for calling attention to this reference.

- $\left|S^{(n)}\right| \leq \bar{q}n$ for all n.[9]

Call a random market *one-to-one* if $q_c = 1$ for all $c \in C^{(n)}$. Call such a market *balanced* if $\left|S^{(n)}\right| = \sum_{c \in C^{(n)}} q_c$ and (p,r)*-unbalanced* if $\left|S^{(n)}\right| = r + p \sum_{c \in C^{(n)}} q_c$.

3 Results

I present a lower bound on core size and describe incentives to manipulate stable matching mechanisms in k-nearest-colleges random markets. These results apply to a broader class of models with preference locality; in Appendix D, I discuss a more general preference locality condition that yields similar results.

3.1 Core size in large, one-to-one markets

Consider a sequence of random markets $\left(\tilde{\Gamma}^{(n)}\right)_{n \in \mathbb{N}}$. For a random market $\tilde{\Gamma}^{(n)}$, let $\alpha(n)$ denote the expected number of colleges with multiple stable allocations and let $\beta(n)$ denote the expected number of students with multiple stable allocations.

Theorem 3.1. *Given a regular sequence of balanced, one-to-one, k-nearest-colleges random markets, there exists $\Delta > 0$ such that:*

- $\liminf_{n \to \infty} \alpha(n)/n > \Delta$.
- $\liminf_{n \to \infty} \beta(n)/n > \Delta$.

Proof. The proof roughly follows that of Theorem 2 of Hassidim *et al.* (2019), who use a similar construction to bound below the fraction of agents with multiple stable allocations, in their matching-with-contracts setting.

Consider two students $s_1, s_2 \in S^{(n)}$ and two colleges $c_1, c_2 \in C^{(n)}$. Let the event $E^{(n)}(s_1, s_2, c_1, c_2)$ denote the case where:

1. College c_1 prefers s_1 to s_2. Formally, $s_1 \succ_{c_1} s_2 \succ_{c_1} \varnothing$.
2. College c_2 prefers s_2 to s_1. Formally, $s_2 \succ_{c_2} s_1 \succ_{c_2} \varnothing$.

[9] Following Ashlagi *et al.* (2017), I drop from the regularity conditions of Kojima and Pathak (2009) the condition that all students be acceptable to all colleges. In these terms, a regular sequence of Kojima–Pathak random markets with all students acceptable to all colleges is what Kojima and Pathak (2009) call a regular sequence of random markets.

3. The only students who find either c_1 or c_2 acceptable are s_1 and s_2. Formally, for all $s \in S^{(n)} \backslash \{s_1, s_2\}$, $\varnothing \succ_s c_1$ and $\varnothing \succ_s c_2$.

4. The colleges that s_1 finds most desirable are c_2 and c_1, in that order. Formally, for all $c \in C^{(n)} \backslash \{c_1, c_2\}$, $c_2 \succ_{s_1} c_1 \succ_{s_1} c$.

5. The colleges that s_2 finds most desirable are c_1 and c_2, in that order. Formally, for all $c \in C^{(n)} \backslash \{c_1, c_2\}$, $c_1 \succ_{s_2} c_2 \succ_{s_2} c$.

Note that, in the event $E^{(n)}(s_1, s_2, c_1, c_2)$, the four agents have two stable allocations: $\{(s_1, c_2), (s_2, c_1)\}$ (the student-optimal allocation), and $\{(s_1, c_1), (s_2, c_2)\}$ (the college-optimal allocation).

The first statement ($\liminf_{n\to\infty} \alpha(n)/n > \Delta$) follows from Lemma 3.2 below, which places a positive lower bound[10] on the probability that a college c_1 is involved in some such $E^{(n)}(s_1, s_2, c_1, c_2)$, and which holds for sufficiently large n.

The second statement ($\liminf_{n\to\infty} \beta(n)/n > \Delta$) then follows by a counting argument: each pair of colleges c_1, c_2 where an event $E^{(n)}$ occurs corresponds to exactly one pair of students s_1, s_2 with multiple stable allocations. □

Remark 2. *The conditions that define $E^{(n)}(s_1, s_2, c_1, c_2)$ are unnecessarily restrictive, in the interest of a cleaner bounds calculation. Theorem 3.1's proof would still hold with condition 3 weakened to require only that s_1 and s_2 are* the most preferable *students (to c_1 and c_2) among those who find c_1 and c_2 acceptable. Similarly, conditions 4 and 5 could be weakened to require only that c_1 and c_2 are, in the the specified orders, the students' most-preferred colleges* among their feasible matches*, and c_1 and c_2 could be any two schools within k places, rather than necessarily adjacent. For experimental estimates of the overall incidence of multiple stable matches in k-nearest-colleges markets, see § 4.*

Lemma 3.2. *Fix an arbitrary $\epsilon > 0$. There exists sufficiently large n such that for each college $c_1 \in C^{(n)}$ the event*

$$E_{c_1}^{(n)} := \bigcup_{(s_1,s_2,c_2)\in S^{(n)}\times S^{(n)}\times C^{(n)}} E^{(n)}(s_1, s_2, c_1, c_2) \tag{3.1}$$

has probability bounded below by $\Delta := \frac{\exp[-k-1]}{4k^2} - \epsilon$.

[10] Specifically, $\Delta = \frac{\exp[-k-1]}{4k^2}$.

Proof. Fix an arbitrary $c_1 \in C^{(n)}$. For different selections of (s_1, s_2, c_2), the events $E^{(n)}(s_1, s_2, c_1, c_2)$ are disjoint. There are $n \cdot (n-1)$ possible selections such that $s_1 \neq s_2$ and $y_{c_2} - y_{c_1} = 1$. Half of these events have zero probability (when $s_2 \succ_{c_1} s_1$). The probability of each of the other events is at least

$$\frac{1}{2} \cdot \left(1 - \frac{k+1}{n-1}\right)^{(n-2)} \cdot \frac{1}{nk} \cdot \frac{1}{nk}, \tag{3.2}$$

where each term in this expression corresponds to an (independent) requirement[11] from the definition of $E^{(n)}(s_1, s_2, c_1, c_2)$. Thus, the probability of the event $E_{c_1}^{(n)}$ is at least $n \cdot (n-1) \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \left(1 - \frac{k+1}{n}\right)^{(n-2)} \cdot \frac{1}{nk} \cdot \frac{1}{nk}$. The limit inferior of this expression as $n \to \infty$ is $\Delta := \frac{\exp[-k-1]}{4k^2}$. □

3.2 Core size in large, unbalanced markets

Again let $\alpha(n)$ denote the expected number of colleges with multiple stable allocations in a sequence of random markets $\left(\tilde{\Gamma}^{(n)}\right)_{n \in \mathbb{N}}$, and let $\beta(n)$ denote the expected number of students with multiple stable allocations.

Theorem 3.3. *Given a $(k, \overline{q})$-regular sequence of (p, r)-unbalanced, k-nearest-colleges random markets, there exists $\Delta > 0$ such that:*

- $\liminf_{n \to \infty} \alpha(n)/n > \Delta$.
- $\liminf_{n \to \infty} \beta(n)/n > \Delta$.

Appendix B presents the proof, which proceeds by a similar construction to Theorem 3.1, modified so that c_1's quota is filled, ensuring that c_2's rejection of s_1 forces c_1 to reject some other student s_2 in turn.

3.3 Incentives to manipulate in large, unbalanced markets

Fix some core-selecting mechanism and consider a sequence of random markets $\left(\tilde{\Gamma}^{(n)}\right)_{n \in \mathbb{N}}$. For a random market $\tilde{\Gamma}^{(n)}$, let $\gamma(n)$ denote the expected number of agents who can improve their assigned outcomes by misreporting their true preferences (when all others report their true preferences).

Corollary 3.4. *Given a $(k, \overline{q})$-regular sequence of (p, r)-unbalanced, k-nearest-colleges random markets, there exists $\Delta > 0$ such that $\liminf_{n \to \infty} \gamma(n)/n > \Delta$.*

[11] To be precise, the second, third, fourth, and fifth requirements, respectively.

Proof. Consider the argument of Theorem B.1. In event $E^{(n)}(s_1, \ldots, s_{q+1}, c_1, c_2)$, the four agents s_1, s_2, c_1, c_2 have two stable allocations: $\{(s_1, c_2), (s_2, c_1)\}$ (the student-optimal allocation), and $\{(s_1, c_1), (s_2, c_2)\}$ (the college-optimal allocation).

Whichever allocation the mechanism assigns, either of the disfavored agents can improve their assigned outcome by instead (mis)reporting that only their optimal match is acceptable. Applying Lemma 3.2 provides a positive lower bound for the probability that any college is involved in such an event, and thus, the expected fraction of agents with incentives to manipulate. □

4 Computational experiments

I present simulation results that complement these theoretical results by demonstrating the realized size of large cores in k-nearest-colleges random markets. My approach follows Ashlagi *et al.* (2017), replicating certain results under their market specifications, then presenting analogous results under models of preferences with locality. For each market specification, I simulate a number of realizations by drawing random preferences independently for each agent, and computing the stable match optimal for each side of the market. I present additional figures in Appendix C.

4.1 Strategic incentives in unbalanced markets

The first experiment illustrates that preference locality can support scope for strategic incentives and attenuate the sharp welfare effect of imbalance[12] in a small market. I specify a k-nearest-colleges random market with 40 colleges, between 20 and 80 students, and $k \in \{3, 5, 10, 20, 40\}$. I simulate 30,000 realizations.

Figure 1 reports the fraction of matched agents who have multiple stable matches; this fraction is small in unbalanced markets under homogeneous ($k = 40$) preferences, but substantial for even substantially unbalanced markets where $k = 3$, 5, or 10.

Figure 2 reports averages across realizations of the matched students' rank of matches under the student-optimal and student-pessimal stable matches. The results for $k = 40$ replicate prior work by Ashlagi *et al.* (2017), who describe the homogeneous case:

[12] *cf.* the sharp welfare effect of imbalance in homogeneous random markets demonstrated by Ashlagi *et al.* (2017).

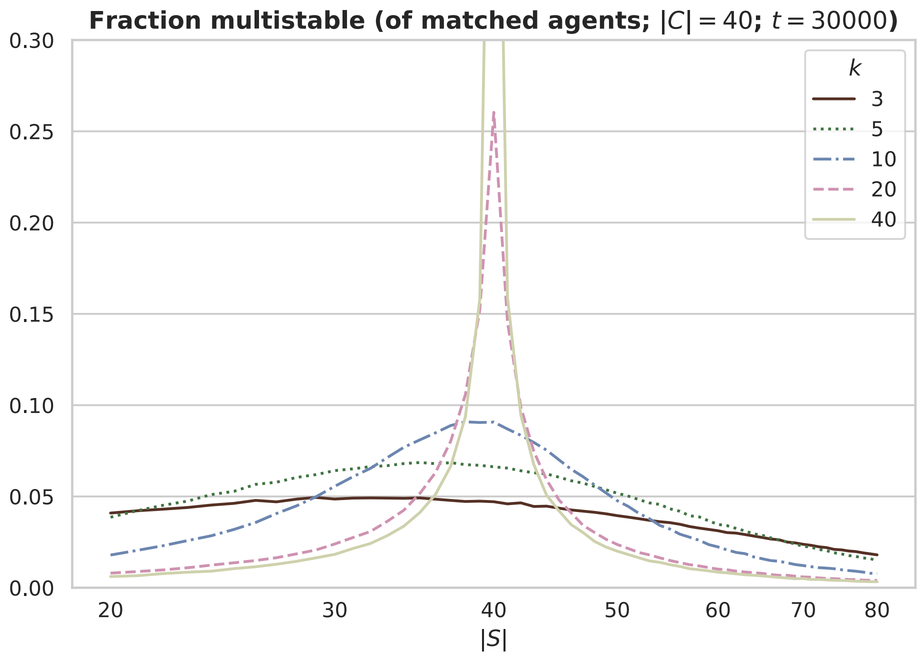

Figure 1: Core size for $|C| = 40$ and $|S|$ from 20 to 80.

> [I]n any unbalanced market, the men's average rank of wives is almost the same under the [men-optimal stable match] and [women-optimal stable match]. When there are fewer men than women (*i. e.*, fewer than 40 men), the men's average rank of wives under any stable matching is almost the same as under [random serial dictatorship], with most men receiving one of their top choices. When there are more men than women in the market, the men's average rank of wives is not much better than 20.5 [of 40], which would be the result of a random assignment.

But under more-localized (*i. e.*, smaller-k) preferences, three differences are apparent:

- The students' welfare gap between optimal and pessimal matches is substantially larger in unbalanced markets.
- The students' welfare gap does not increase sharply as markets approach balance.
- Students' absolute welfare (under any stable match) depends less sharply on the imbalance.

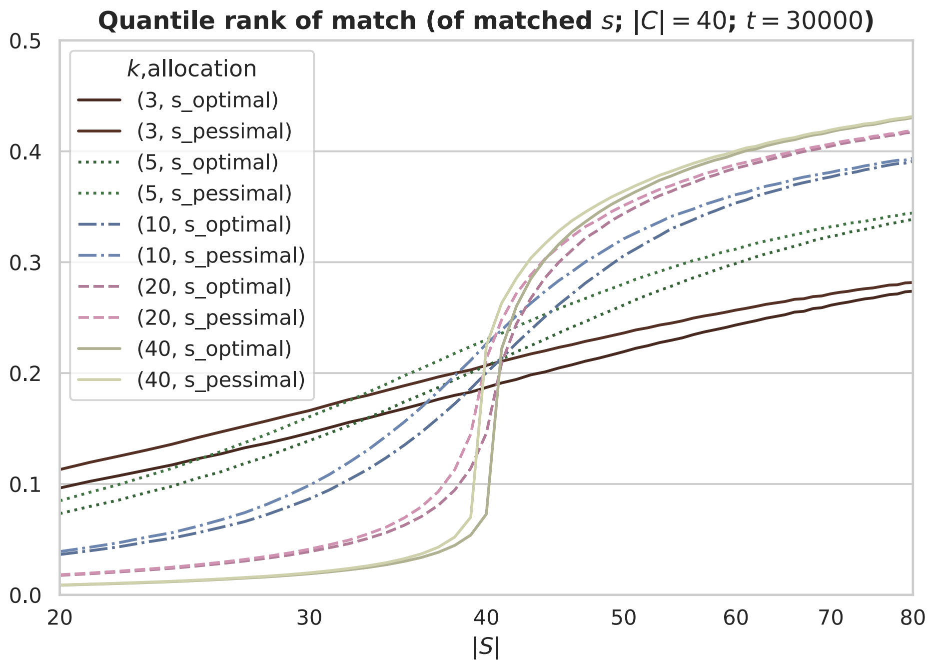

Figure 2: Matched students' quantile rank of matches for $|C| = 40$.

Colleges' welfare gaps and absolute welfare respond similarly to preference locality; see Figures 5 and 6 in Appendix C.

4.2 Core size and allocation outcomes in large markets

I also present results of simulations of large unbalanced matching markets under various specifications. Figure 3 reports the fraction of matched agents who have multiple stable matches and matched students' average rank of matches (across 3,000 realizations) in a market with 400 colleges, between 200 and 800 students, and preferences given by the k-nearest-colleges model for $k \in \{3, 10, 30, 100, 400\}$.

In general, as in the smaller specification, the size of the core and the strategic incentives do decrease as the market becomes unbalanced, but much more slowly under more-localized (*i. e.*, smaller-k) preferences than under the homogeneous specification. As an example: in a market with 440 students, 400 colleges, and $k = 30$, $> 5\%$ of matched students have more than one stable match, and among students with more than one stable match, the average rank difference between their optimal and pessimal match is > 7 (*i. e.*, nearly a quarter of their preference list length). In the

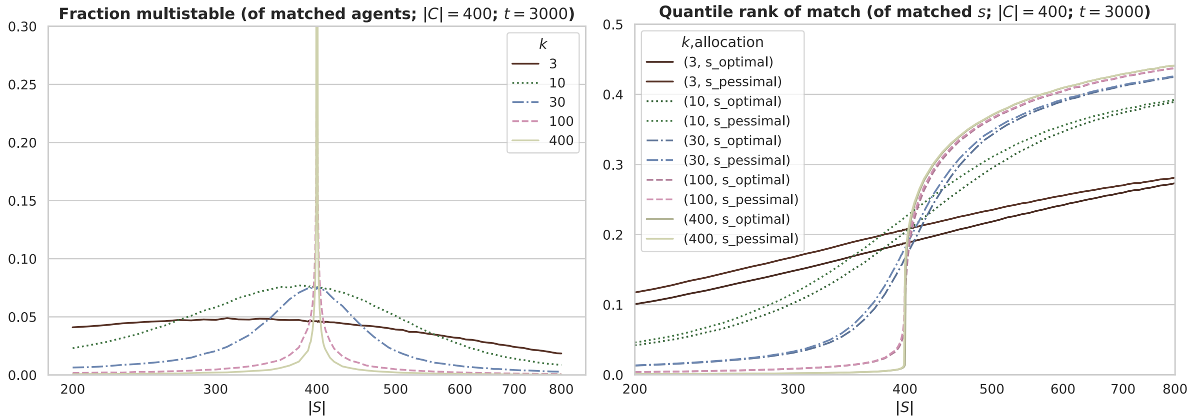

Figure 3: Core size and matched students' quantile rank of matches for $|C| = 400$.

homogeneous specification, the incentives are only an eighth as large. In a more unbalanced market, the effect is even more stark—with 500 students, 400 colleges, and $k = 10$, $> 5\%$ of matched students have more than one stable match, but with $k = 400$, $< 0.24\%$ do.

I present further simulation results for markets with $q|C| \in \{400, 1000, 4000\}$ college-seats, and for $q \in \{1, 4, 10\}$, in Figures 7–9 in Appendix C. The gap in student welfare between the student-optimal and student-pessimal allocations remains apparent, and the fraction of agents with more than one stable match fails to vanish, as the market size grows.

4.3 Localized versus incomplete homogeneous preferences

It is apparent to visual inspection that in any of the above specifications, matched students' welfare (under any stable match) depends less sharply on the cross-side imbalance under more-localized (*i. e.*, lower-k) preferences than under homogeneous preferences. This agrees with simulation results of Ashlagi *et al.* (2017) on incomplete homogeneous student preferences (in which students find some fraction of colleges unacceptable, and otherwise draw preferences uniformly), suggesting that the welfare-flattening effect could be driven by selectivity rather than preference structure. I confirm this hypothesis with simulation results, but find that substantial welfare gaps are *not* supported under incomplete homogeneous preferences.

By setting the probability that a student s finds a college c acceptable to $k/|C|$, I specify incomplete homogeneous preferences under which students

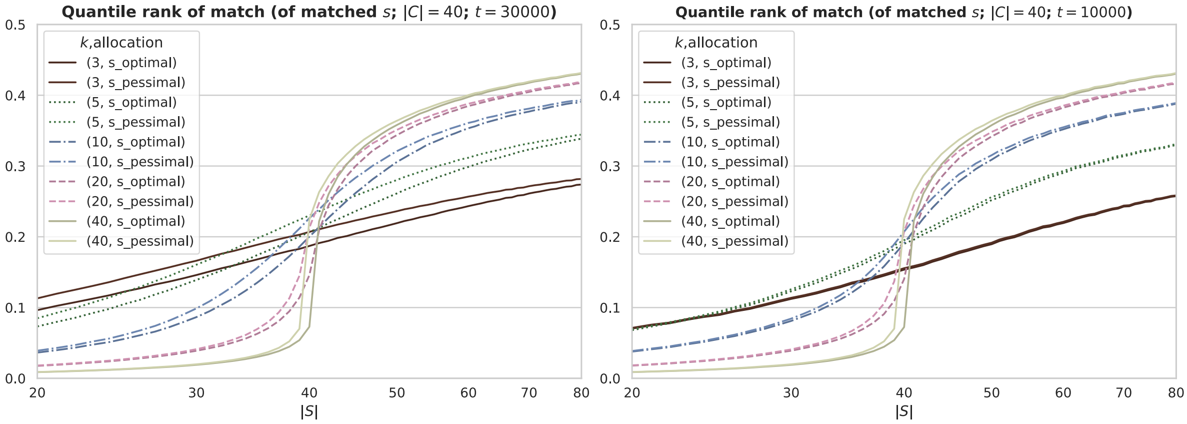

Figure 4: Matched students' quantile rank of matches under k-nearest-colleges preferences (left) and similarly-selective homogeneous preferences (right). Plot at left reproduces Figure 2.

are on average as selective as in a k-nearest-colleges model. Figure 4 reports averages across realizations of the matched students' rank of matches under such a preference model, and under a k-nearest-colleges model.

While the absolute welfare effect of student selectivity is comparable—attenuating the short-side advantage similarly in both specifications—market power in unbalanced markets is only supported under the preference-locality specification. A minor effect is visible in low-k specifications where students are assigned slightly worse-ranked matches under the preference-locality specification, regardless of imbalance size. This suggests, intuitively, that certain forms of localized same-side correlation in students' preferences can be disadvantageous to students' welfare, even holding selectivity constant and setting colleges' relative popularities to be equal.

Similar results are visible in simulations of $|C| = 400$ and $|C| = 1000$ markets, presented in Figure 10 in Appendix C.

5 Discussion

It is beyond the scope of the present work to present a comprehensive structural theory of matching-market dynamics. However, I discuss some ways in which locality interacts with other structural properties of matching markets, and their implication for matching dynamics, presenting some questions for further investigation.

5.1 Related work: matching markets

The proofs presented here use techniques for constructively demonstrating market power (in the large-market limit) similar to those presented by Hassidim *et al.* (2019) in a setting of matching with contracts. I apply these techniques instead to the setting of matching (a) without contracts but (b) with locality in preference structure. This contrasts previous work on matching without contracts that analyzes homogeneous models of large matching markets, primarily through stochastic analysis of rejection chains, in one-to-one (Immorlica and Mahdian (2005)), many-to-one (Kojima and Pathak (2009)), and unbalanced (Ashlagi *et al.* (2017)) settings.[13]

This prior work has generally suggested that homogeneous large markets exhibit vanishing market power (Immorlica and Mahdian (2005); Kojima and Pathak (2009)) and in many cases small cores (Ashlagi *et al.* (2017)), and that these features are generally strengthened by correlation among preferences.[14] The two examples of non-vanishing market power in the literature (excluding the knife-edge case of balanced markets) are the Immorlica and Mahdian (2005) paired-colleges construction[15], and the Hassidim *et al.* (2019) case of college admissions with financial aid contracts. The present work explains that both cases are essentially driven by localized preference structures (localized to pairs of colleges or to sets of contracts with the same college, respectively), and proposes the first plausible structural model of preferences that supports market power in the absence of contracts.

5.2 Related work: network structure

My approach to modeling preference structure draws from the computer science literature on network structure in large graphs. Early analysis of large homogeneous graphs (Gilbert (1959); Erdős and Rényi (1960)) was

[13]Kojima and Pathak (2009) also consider the union of a bounded number of homogeneous models, to represent agent types. However, with a bounded number of agent types, this model exhibits vanishing locality, as discussed in Appendix D.

[14]The primary form of correlation in the referenced literature is concentration of global popularity. Ashlagi *et al.* (2017) consider a multiparameter model of correlated preference structure, including a parameter for locality. However, their locality parameter induces alignment between sides of the market, simultaneously shrinking the possible difference between the student-optimal and college-optimal stable matches. Unsurprisingly, they find that increasing this parameter does little to increase the size of the core (though it does cause a slight increase under sufficiently unbalanced market specifications). By contrast, I present a model of preference locality *without* cross-side preference alignment, and find large cores supported under certain market specifications.

[15]See Remark 1 above, and Remark 2.7 in Immorlica and Mahdian (2005).

conducted via combinatorial techniques similar to the existing literature on large matching markets. However, the contemporary network-structure literature has turned to non-homogeneous models to model structural features empirically observed in real-world graphs—*e. g.*, locality (Watts and Strogatz (1992); Leskovec *et al.* (2009)); the small-world property (Leskovec *et al.* (2005); Bordino *et al.* (2008)); power-law distribution of node degree (Barabási and Albert (1999); Aiello *et al.* (2000)); self-similar subnetwork structure (Chakrabarti *et al.* (2004)); and others (Boldi *et al.* (2011); Yang and Leskovec (2014)).

The specific preference-locality structure I consider is inspired in form by spatially-structured locality models for network graphs (Watts and Strogatz (1992)). While I do not consider the analogues of more-sophisticated generative models in this work, they do suggest that locality (in forms that support market power in the large-market limit) can exist under generative models of a more natural flavor than the synthetic model presented in the present work. In Appendix D, I state a generalized locality property sufficient to yield my main results and suggest common generative network models from the network-structure literature under which it obtains.

Market agents forming preferences over potential matches are plausibly influenced by similar processes as agents forming ties in a network graph. On this basis, I propose that further work could fruitfully build on the network structure literature to better model the structures of matching markets among agents embedded in the actual world, and explore the welfare implications and strategic incentives that arise from them.

5.3 Small cores in large markets

In apparent contrast to the present work, some empirical studies of matching markets have found small cores in large markets. For example, Roth and Peranson (1999) examined the National Resident Matching Program (NRMP)'s market for medical residency positions from 1993 to 1996 and found that in a market with roughly 20,000 applicants and potential positions, only about 0.1% of residents were assigned different matches by primarily resident-proposing and program-proposing mechanisms.[16] This is

[16]Neither the baseline NRMP mechanism nor the mechanism redesigned by Roth and Peranson (1999) were simple deferred-acceptance mechanisms, so this comparison is not completely within the theoretical framework of the present work. Nevertheless, the mechanisms in question were both based heavily on deferred-acceptance mechanisms, and did differ in whether residents or programs proposed in the primary stage. Comparable results obtained when the same authors investigated the fraction of hospitals with incentives to manipulate the redesigned, applicant-proposing mechanism.

a significant divergence from the roughly 4% that might be predicted by some models presented in the present work.

These small cores are not due to structural variations in the NRMP match—in the same work, the authors presented five years of data on a matching market for thoracic surgery residents with roughly 175 applicants (130 positions) and no match variations, in which $< 0.5\%$ of residents had more than one stable match; some models of the present work might suggest rates ten times that, given localized preferences. Similarly, Kojima *et al.* (2013) examined a matching market for clinical psychologists (with roughly 3,000 applicants[17] and 2,700 positions) from 1999 to 2007 and found that roughly 0.2% of residents had more than one stable match. Pathak and Sönmez (2008) examined two years of matching markets for primary- and secondary-school admissions (with roughly $3,000$ applicants at each level per year) in the Boston Public Schools and found five instances of applicants with multiple stable matches.

These observations suggest that agents in these markets have preferences which are not well-modeled by locality alone. For example, Kojima *et al.* (2013) found a concentration of program popularities in the clinical psychology match, with a small number of programs receiving as many as eight times the number of first-place rankings that would be predicted by uniform draws. The authors further note that "these are preferences stated after interviews have been conducted, so [they do] not preclude the possibility that there are popular programs that receive many applications but only interview a small subset of applicants", and further that "an applicant typically ranks a program only after she interviews at the program, and each applicant receives and can travel to only a limited number of interviews."

However, Kojima *et al.* (2013) did find locality structure in the clinical psychology match—in their sample, half of single applicants rank programs in at most two of eleven geographical regions. In addition to national geography, location of commuter-student schools within a city or applicants' preferences for institutional features such as operational style or specialty focus might induce locality among applicant preferences. It is intuitive that such factors would exist in many of the settings encountered in this literature.

These structural features of agents' preferences can have complex effects on matching dynamics such as core size and cross-side welfare distribution, in light of market size and cross-side imbalance. Full analysis of these in-

[17] The cited authors removed pairs of applicants registered as couples (roughly 19 per year) for the cited experiment.

teractions are beyond the scope of the present work. It is similarly beyond the scope of the present work to analyze the extent to which apparently small cores in residency matches, or other markets of interest, are created by successful strategic manipulations by residency programs (including manipulation of interviewing capacity)[18].

5.4 Structure and core size

To briefly summarize the directional effects that structural factors can have on match dynamics:

- Cross-side imbalance decreases core size and distributes welfare to favor the shorter side (Ashlagi *et al.* (2017)).
- Concentration of general popularity on one side of the market decreases core size and attenuates the welfare effect of cross-side imbalance.
- Selectivity[19] decreases core size (Immorlica and Mahdian (2005); Kojima and Pathak (2009)) and attenuates the welfare effect of cross-side imbalance (Ashlagi *et al.* (2017)).
- Preference locality increases core size (as per the present work).
- Larger markets amplify the welfare effect of cross-side imbalance (holding imbalance ratio constant), and decrease core size in the presence of selectivity effects or cross-side imbalance (as presented in § 4 and Appendix C).

[18]Mongell and Roth (1991) study a matching market for membership in undergraduate social organizations and propose that empirically-observed but theoretically surprising matching dynamics such as stability might be caused by the accepting side of the market manipulating preference lists to obtain the stable match optimal for that side.

[19]The length of *submitted* rank-order lists may not be an accurate indication of selectivity in settings where agents on either side face costs to search, or are actively engaging in strategic manipulation strategies. In such cases, reported preferences may misleadingly suggest a small core by falsely excluding matches that would be stable with respect to true preferences, but do not appear among reported preferences.

Limitations on preference expression (whether exogenously imposed or endogenous from search costs) can also create particularly misleading results in a setting like that in Shorrer (2020), where both sides are (imperfectly) vertically differentiated and proposing agents face uncertainty about their relative standing. In such a setting, students' optimal application portfolios include colleges with diversified quality levels, and *ex post* their apparent stable matches will be largely determined by their own realized quality level. Other colleges of similar quality levels (which are, *ex post*, likely candidates for stable matches) are disproportionately *unlikely* to appear in an optimally-diversified portfolio, falsely excluding most potential matches that are stable with respect to true preferences.

Cross-side imbalance, concentration of popularity, and selectivity need not be present globally to have their effects on the local core—and so may be relevant on the local level in markets with sufficient locality in preferences. Consider a model of preference structure under which medical residents agree on intra-city ranking of hospitals, but have different preferences on which city to work in. If the effective number of hospitals per city is large relative to the effective number of cities, then this market will act similarly to a market with common residents' preferences, and will exhibit a small core and an attenuated welfare effect of cross-side imbalance. Similarly, a highly-segmented market with varying levels of cross-side imbalance will have a small local core and strong cross-side welfare effect in a highly-unbalanced segment, regardless of the global cross-side balance.

As locality in networks can be fractal in nature (Chakrabarti *et al.* (2004); Leskovec *et al.* (2009)), the effects of these structural factors may be mediated across scales of locality by the structure of agents' preferences. I hope that future work can explore how these effects propagate through the network topologies induced by empirically-observed preference structures to affect dynamics of distributional welfare and incentive-compatibility for all participants.

5.5 Conclusion

This work presents an opportunity at the intersection of two increasingly relevant topics—design of mechanisms for large matching markets and structural properties of large networks—to better understand the real-world settings addressed by the market-design literature. By considering the matching-market analogue of a simple model of network structure, I found welfare and incentive dynamics not previously found in homogeneous market models. Unlike in homogeneous market models, these effects fail to vanish in the large-market limit, and in simulation experiments, they are substantial in magnitude.

As the stability and incentive-compatibility of matching-market mechanisms is understood to be of primary importance to the success of matching marketplaces, these results raise the potential for concern in markets not known to have empirically small cores, but where preferences may plausibly exhibit locality structure. For markets with empirically-observed very small cores, these results suggest that the small core is not induced by market size or imbalance alone, but due to other structural factors not yet investigated in the literature.

I anticipate scope for future work that (a) uses tools from the network-structure literature to characterize the structures of preferences empirically found in large markets and (b) applies the models and techniques techniques from the prior literature and the present work directly to better-tailored structural models of markets of interest.

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A Formal definitions excluded from main text

A.1 Matching markets

Given a set of colleges C and a set of students S (together *agents*), each college c has a complete strict preference relation $\succ_c$ over the subsets of students 2^S and each student s has a complete strict preference relation $\succ_s$ over colleges C and the outcome of being unmatched (denoted $\varnothing$). A student s is *acceptable* to a college c if $\{s\} \succ_c \varnothing$ and a college c is *acceptable* to a student s if $c \succ_s \varnothing$. A *matching market* is a tuple of colleges, students, and agent preferences.

For a college c and a *quota* q_c, the preference relation $\succ_c$ is *responsive with quota* q_c if the ranking of a student is independent of their colleagues, and all sets of students exceeding quota q_c are unacceptable (see Roth (1985) for further discussion). I consider only responsive college preferences in the present work.[20] Furthermore, I abuse notation and write $s \succ_c \varnothing$ and $s_1 \succ_c s_2$ to indicate $\{s\} \succ_c \varnothing$ and $\{s_1\} \succ_c \{s_2\}$ when discussing colleges' preferences with respect to individual students.

A *matching* is a mapping μ on $C \cup S$ that associates colleges to disjoint sets of students, and students to the corresponding college or unmatched outcome:

- For $c \in C$, $\mu(c) \in 2^S$.
- For $s \in S$, $\mu(s) \in C \cup \varnothing$.
- For $c, s \in C \times S$, $s \in \mu(c) \iff \mu(s) = c$.

A matching μ is *blocked* by a college-student pair c, s if:

- s prefers c to their match ($c \succ_s \mu(s)$).
- Either c has a vacancy and finds s acceptable ($|\mu(c)| < q_c$ and $s \succ_c \varnothing$), or c prefers s to some other matched student ($\exists s' \in \mu(c) : s \succ_c s'$).

A matching is *stable* if it is unblocked, each college c is matched to a number of acceptable students no more than q_c, and each matched student is matched to an acceptable college.

[20] Kojima and Pathak (2009) note that every responsive preference relation corresponds to an additive utility function over students, providing an intuitive justification for this restriction.

B Proofs excluded from main text

B.1 Core size in large, unbalanced markets

Consider a sequence of random markets $\left(\tilde{\Gamma}^{(n)}\right)_{n\in\mathbb{N}}$. For a random market $\tilde{\Gamma}^{(n)}$, let $\alpha(n)$ denote the expected number of colleges with multiple stable allocations and let $\beta(n)$ denote the expected number of students with multiple stable allocations.

Theorem B.1. *Given a $(k, \overline{q})$-regular sequence of (p, r)-unbalanced, k-nearest-colleges random markets, there exists $\Delta > 0$ such that:*

- $\liminf_{n\to\infty} \alpha(n)/n > \Delta$.
- $\liminf_{n\to\infty} \beta(n)/n > \Delta$.

Proof. Consider two colleges $c_1, c_2 \in C^{(n)}$ and students $s_1, \ldots, s_{q+1} \in S^{(n)}$, where $q := q_{c_1}$. Let the event $E^{(n)}(s_1, \ldots, s_{q+1}, c_1, c_2)$ denote the case where:

1. College c_1 ranks s_2 last of $s_1, \ldots, s_{q+1}$. Formally, for all $s \in \{s_1, s_3, \ldots, s_{q+1}\}$, $s \succ_{c_1} s_2 \succ_{c_1} \varnothing$.
2. College c_2 ranks s_2 higher than s_1. Formally, $s_2 \succ_{c_2} s_1 \succ_{c_2} \varnothing$.
3. The only students who find either c_1 or c_2 acceptable are $s_1, \ldots, s_{q+1}$. Formally, for all $s \in S^{(n)} \backslash \{s_1, \ldots, s_{q+1}\}$, $\varnothing \succ_s c_1$ and $\varnothing \succ_s c_2$.
4. The colleges that s_1 finds most desirable are c_2 and c_1, in that order. Formally, for all $c \in C^{(n)} \backslash \{c_1, c_2\}$, $c_2 \succ_{s_1} c_1 \succ_{s_1} c$.
5. The colleges that s_2 finds most desirable are c_1 and c_2, in that order. Formally, for all $c \in C^{(n)} \backslash \{c_1, c_2\}$, $c_1 \succ_{s_2} c_2 \succ_{s_2} c$.
6. The college that $s_3, \ldots, s_{q+1}$ find most desirable is c_1. Formally, for all $s \in \{s_3, \ldots, s_{q+1}\}$ and $c \in C^{(n)} \backslash \{c_1\}$, $c_1 \succ_s c$.

Note that, in the event $E^{(n)}(s_1, \ldots, s_{q+1}, c_1, c_2)$, the agents have two stable allocations: $\{(s_1, c_2), (s_2, c_1), (s_3, c_1), \ldots, (s_{q+1}, c_1)\}$ (the student-optimal allocation), and $\{(s_1, c_1), (s_2, c_2), (s_3, c_1), \ldots, (s_{q+1}, c_1)\}$ (the college-optimal allocation).

The first statement ($\liminf_{n\to\infty} \alpha(n)/n > \Delta$) follows from Lemma B.2 below, which places a positive lower bound[21] on the probability that a college

[21] specifically, $\Delta = \frac{\min(1,p)^{(\overline{q}+1)} \exp[-p\overline{q}(k+1)]}{4k^2 \overline{q}!}$, or more strongly $\Delta = \frac{p \cdot \exp[-p\overline{q}(k+1)]}{4k^2}$ when $\liminf_{n\to\infty} \left|S^{(n)}\right|/n > 1$, as demonstrated in Lemma B.3

c_1 is involved in some such $E^{(n)}(s_1, \ldots, s_{q+1}, c_1, c_2)$, and which holds for sufficiently large n.

The second statement ($\liminf_{n\to\infty} \beta(n)/n > \Delta$) then follows by a counting argument: each pair of colleges c_1, c_2 where an event $E^{(n)}$ occurs corresponds to exactly one pair of students s_1, s_2 with multiple stable allocations. $\square$

Lemma B.2. *Fix an arbitrary $\epsilon > 0$. There exists sufficiently large n such that for each college $c_1 \in C^{(n)}$ the event*

$$E_{c_1}^{(n)} := \bigcup_{(s_1,\ldots,s_{q+1},c_2)\in\left(S^{(n)}\right)^{\times(q+1)}\times C^{(n)}} E^{(n)}(s_1, \ldots, s_{q+1}, c_1, c_2) \tag{B.1}$$

has probability bounded below by $\Delta := \frac{\min(1,p)^{(\overline{q}+1)}\exp[-p\overline{q}(k+1)]}{4k^2\overline{q}!}$.

Proof. Fix an arbitrary $c_1 \in C^{(n)}$. For different selections of $(s_1, \{s_2, \ldots, s_{q+1}\}, c_2)$, the events $E^{(n)}(s_1, \ldots, s_{q+1}, c_1, c_2)$ are disjoint. There are $\left|S^{(n)}\right| \cdot \left({}_{(|S^{(n)}|-1)}C_q\right)$ possible selections of distinct $(s_1, \{s_2, \ldots, s_{q+1}\})$ and $y_{c_2} - y_{c_1} = 1$. Let s_2 be the c_1-dispreferred student among $\{s_2, \ldots, s_{q+1}\}$ without loss of generality. The probability of each possible $E^{(n)}(s_1, \ldots, s_{q+1}, c_1, c_2)$ is at least

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \left(1 - \frac{k+1}{n-1}\right)^{(|S^{(n)}|-q-1)} \cdot \frac{1}{nk} \cdot \frac{1}{nk} \cdot \left(\frac{1}{n}\right)^{(q-1)}, \tag{B.2}$$

where each term in this expression corresponds to an (independent)[22] requirement from the definition of $E^{(n)}(s_1, \ldots, s_{q+1}, c_1, c_2)$. Thus, the probability of the event $E_{c_1}^{(n)}$ is at least

$$\Delta_q^{(n)} := \left|S^{(n)}\right| \cdot \left(\prod_{i=1}^{q} \frac{\left|S^{(n)}\right| - 1 - i}{i}\right) \cdot \frac{\left(1 - \frac{k+1}{n-1}\right)^{(|S^{(n)}|-q-1)}}{4 \cdot nk \cdot nk \cdot n^{(q-1)}}. \tag{B.3}$$

Let $\overline{Q} := \limsup_{n\to\infty} \left|S^{(n)}\right|/n$ and $\underline{Q} := \liminf_{n\to\infty} \left|S^{(n)}\right|/n$; then

$$\liminf_{n\to\infty} \Delta_q^{(n)} \geq \frac{\underline{Q}^{(q+1)} \cdot \exp\left[-\overline{Q}(k+1)\right]}{4k^2 q!}. \tag{B.4}$$

[22] To be precise, the first, second, third, fourth, fifth, and sixth requirements, respectively, where the first requirement requires only $s_1 \succ_{c_1} s_2$, given that s_2 is the c_1-dispreferred student among $\{s_2, \ldots, s_{q+1}\}$.

So, noting $\overline{Q} \leq p\overline{q}$ and $\underline{Q} \geq p$, conclude

$$P\left[E_{c_1}^{(n)}\right] \geq \Delta := \frac{\min(1,p)^{(\overline{q}+1)} \exp[-p\overline{q}(k+1)]}{4k^2\overline{q}!} \tag{B.5}$$

for sufficiently large n. □

Lemma B.3. *Fix an arbitrary $\epsilon > 0$. If $\underline{Q} > 1$, then there exists sufficiently large n such that for each college $c_1 \in C^{(n)}$ the event $E_{c_1}^{(n)}$ has probability bounded below by $\Delta := \frac{p \cdot \exp[-p\overline{q}(k+1)]}{4k^2}$.*

Proof. If $\underline{Q} > 1$, then $\left|S^{(n)}\right| > n + \overline{q} + 1$ for sufficiently large n. Then $\left({}_{(|S|-1)}C_q\right) > n^q$, so

$$\liminf_{n\to\infty} \Delta_q^{(n)} \geq \frac{\underline{Q} \cdot \exp\left[-\overline{Q}(k+1)\right]}{4k^2} \tag{B.6}$$

and

$$P\left[E_{c_1}^{(n)}\right] \geq \Delta := \frac{p \cdot \exp[-p\overline{q}(k+1)]}{4k^2} \tag{B.7}$$

for sufficiently large n. □

C Figures excluded from main text

This appendix presents figures that extend § 4, reporting average core size and welfare gap size observed in market simulations for various market specifications. Plots are labeled with number of realizations observed (denoted t).

Figures 5 and 6 extend Figures 1–3 and report on one-to-one k-nearest-colleges markets of size $|C| \in \{40, 400, 1000, 4000\}$. Core sizes and welfare gaps remain substantial in unbalanced markets for small k, even as $|C|$ becomes large.

Figures 7–9 report on many-to-one k-nearest-colleges markets of $q \in \{1, 4, 10\}$ and $|C| \in \{400, 1000, 4000\}$. While matched students' average welfare gaps shrink with increased q, gaps remain apparent to visual inspection and the fraction of colleges with incentive to manipulate remains substantial for small k.

Figure 10 extends Figure 4 and compare k-nearest-colleges markets of $|C| \in \{40, 400, 1000\}$ to markets with similarly-selective but homogeneous students. As in the $|C| = 40$ case discussed in § 4.3, the sharp welfare advantage of the short side of the market is attenuated and smoothed by selectivity in both models, though incentives to manipulate are supported only in the k-nearest-colleges model.

C.1 Strategic incentives in unbalanced markets

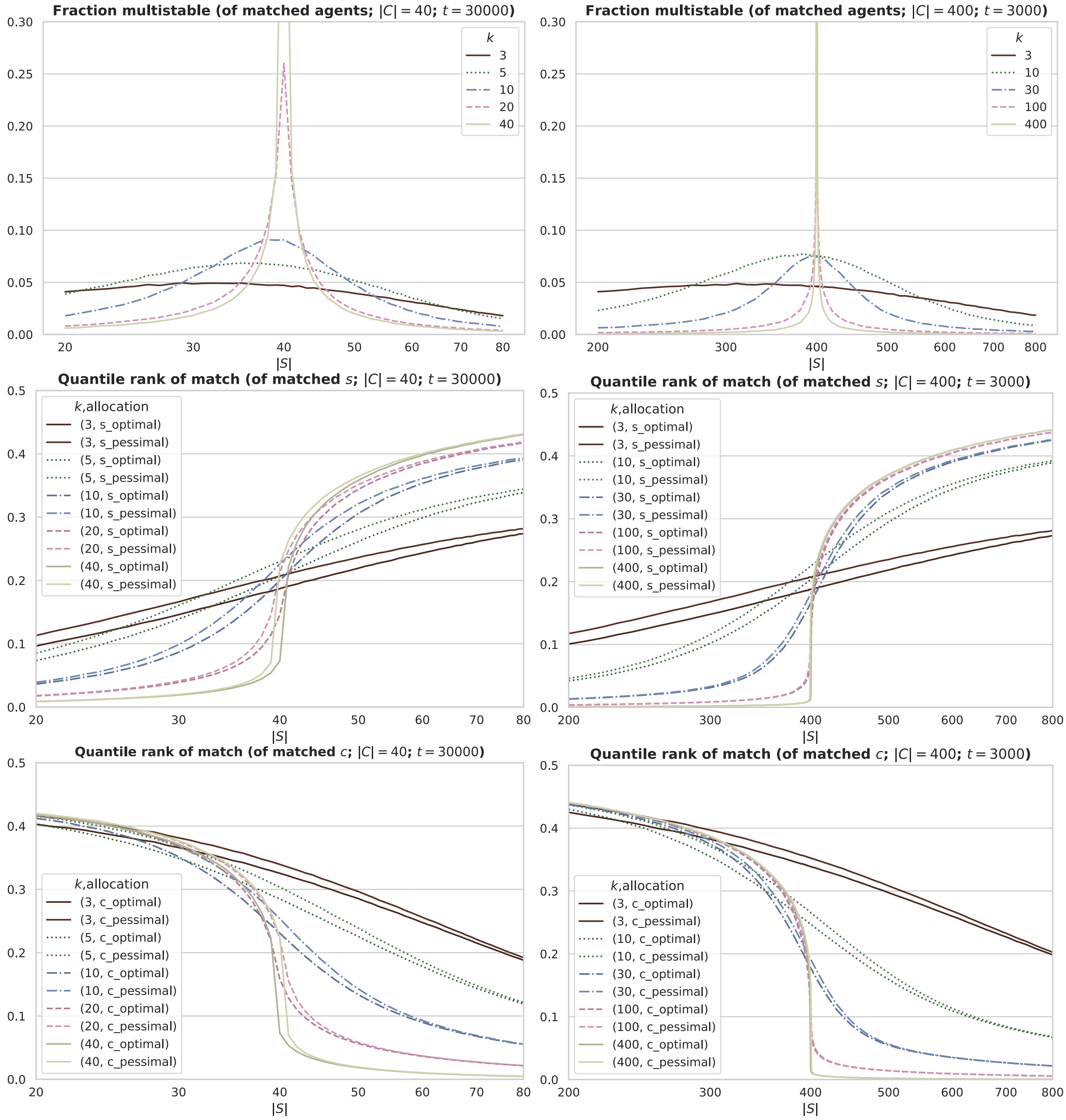

Figure 5: Core size, matched students'/colleges' quantile rank of matches for $|C| = 40$ and $|C| = 400$ under k-nearest-colleges preferences. Plots at top reproduce Figures 1, 2, and 3.

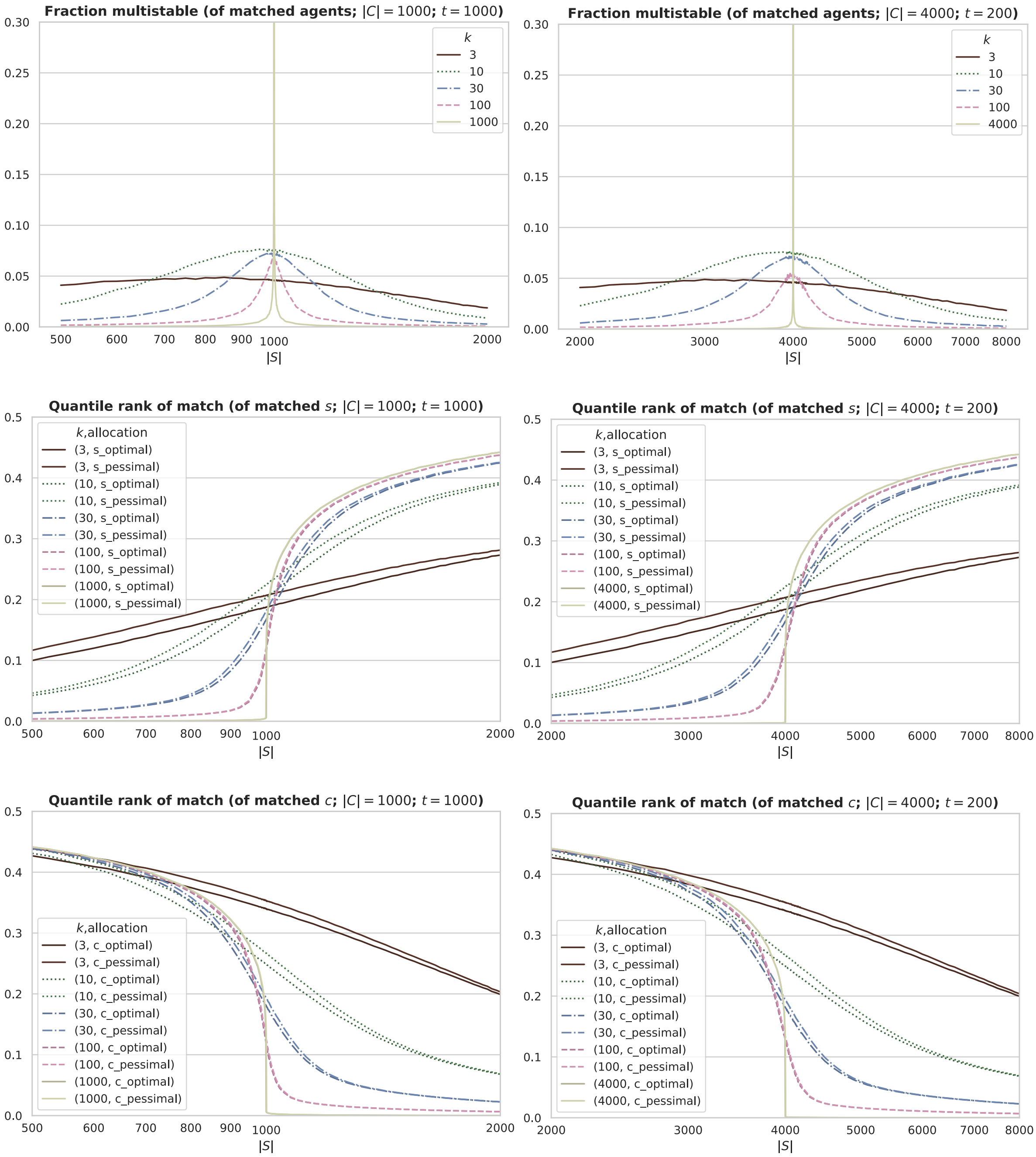

Figure 6: Core size, matched students'/colleges' quantile rank of matches for $|C| = 1000$ and $|C| = 4000$ under k-nearest-colleges preferences.

C.2 Core size and allocation outcomes in large markets

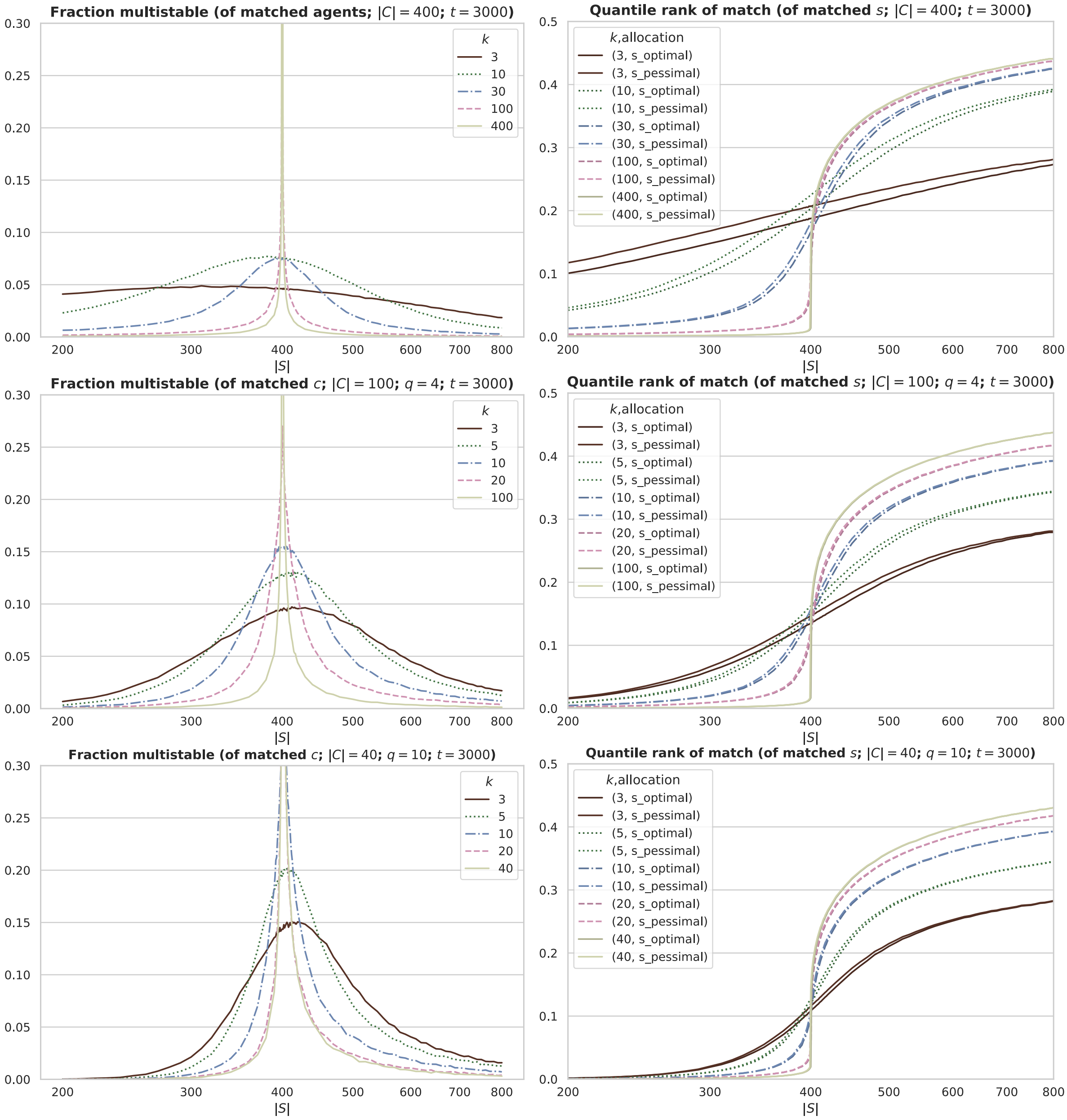

Figure 7: Core size and matched students' quantile rank of matches for $q \in \{1, 4, 10\}$ and $q|C| = 400$ under k-nearest-colleges preferences. Plots at top reproduce Figure 3.

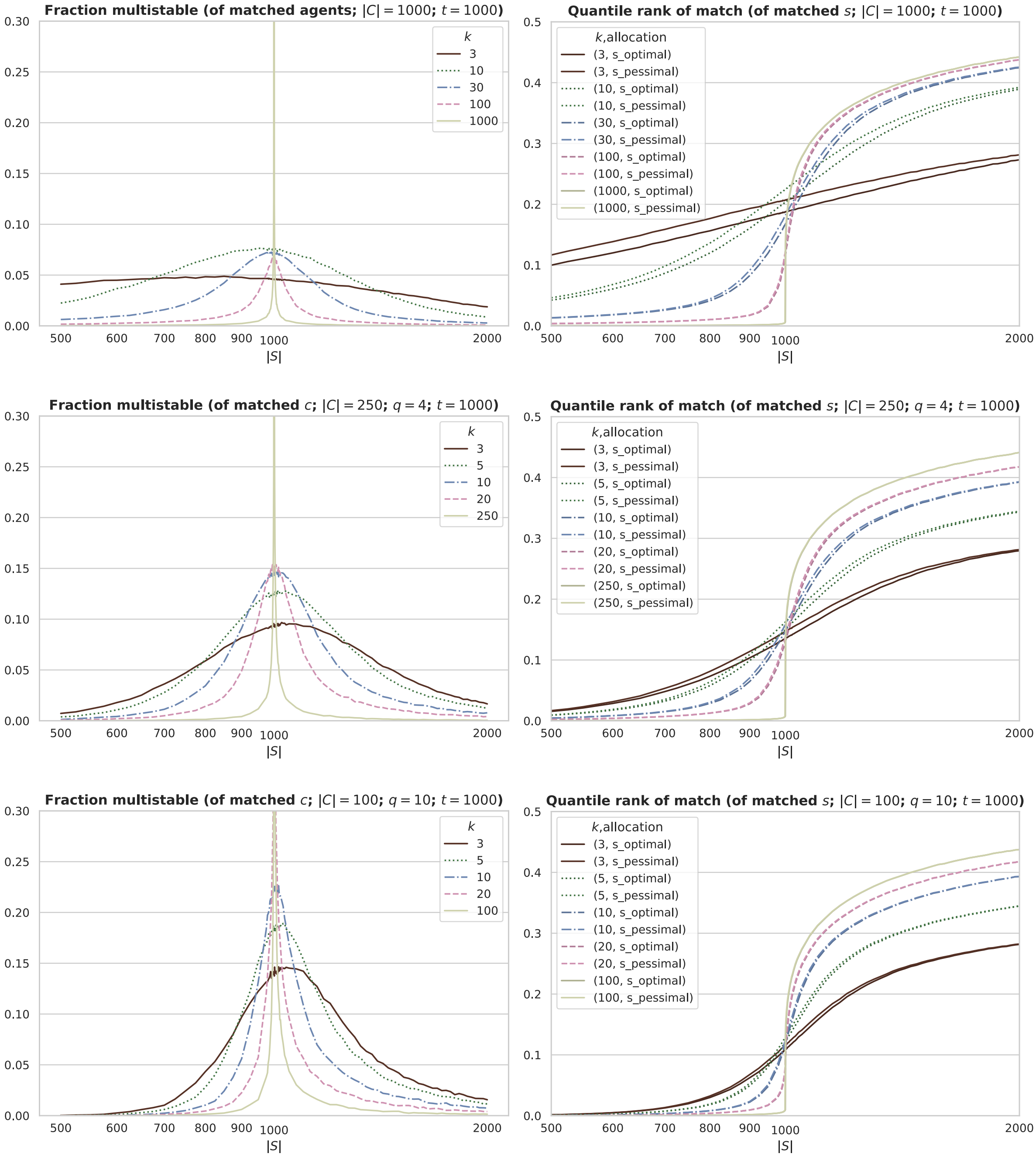

Figure 8: Core size and matched students' quantile rank of matches for $q \in \{1, 4, 10\}$ and $q|C| = 1000$ under k-nearest-colleges preferences. Plots at top reproduce parts of Figure 6.

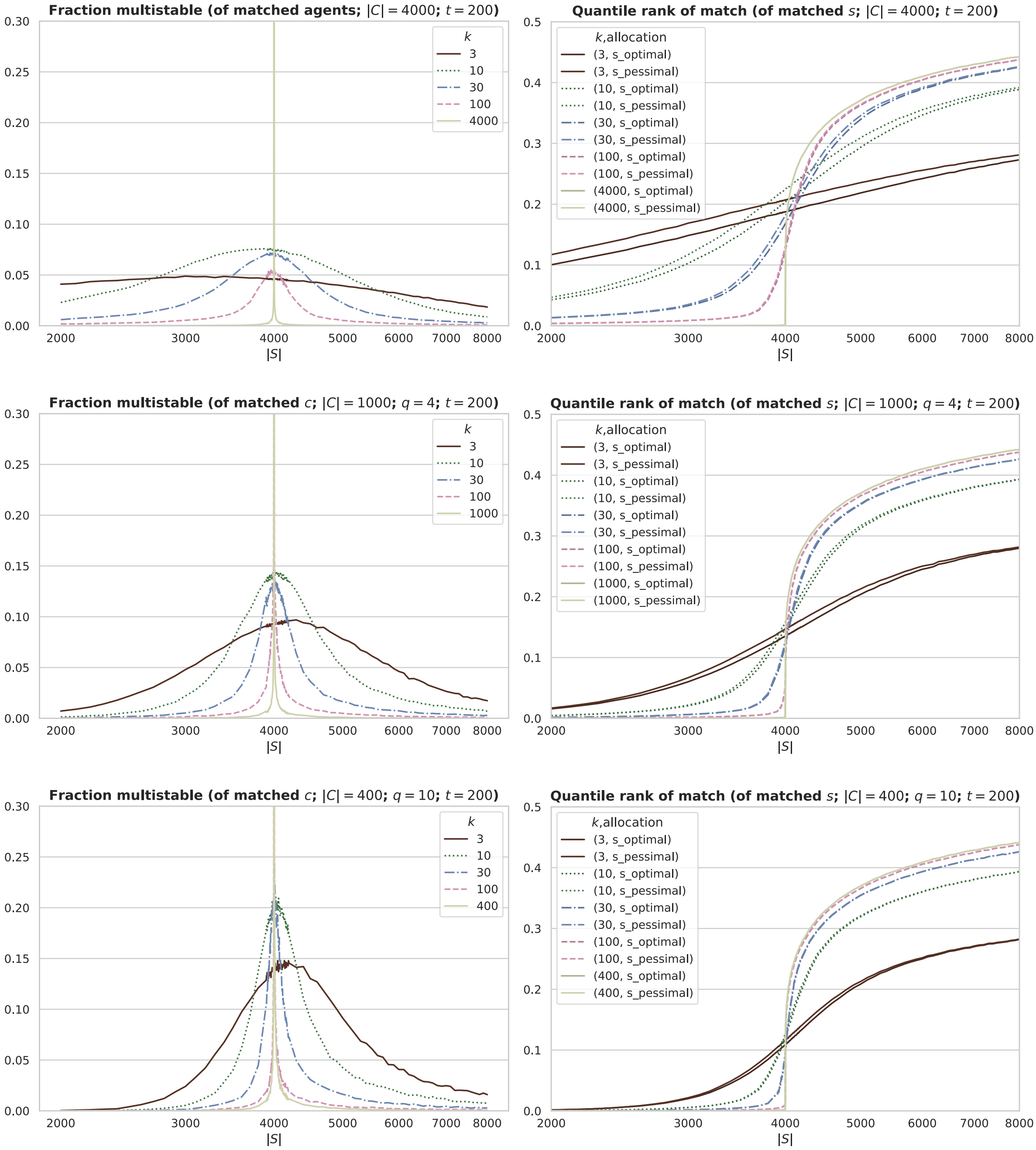

Figure 9: Core size and matched students' quantile rank of matches for $q \in \{1, 4, 10\}$ and $q|C| = 4000$ under k-nearest-colleges preferences. Plots at top reproduce parts of Figure 6.

C.3 Localized versus incomplete homogeneous preferences

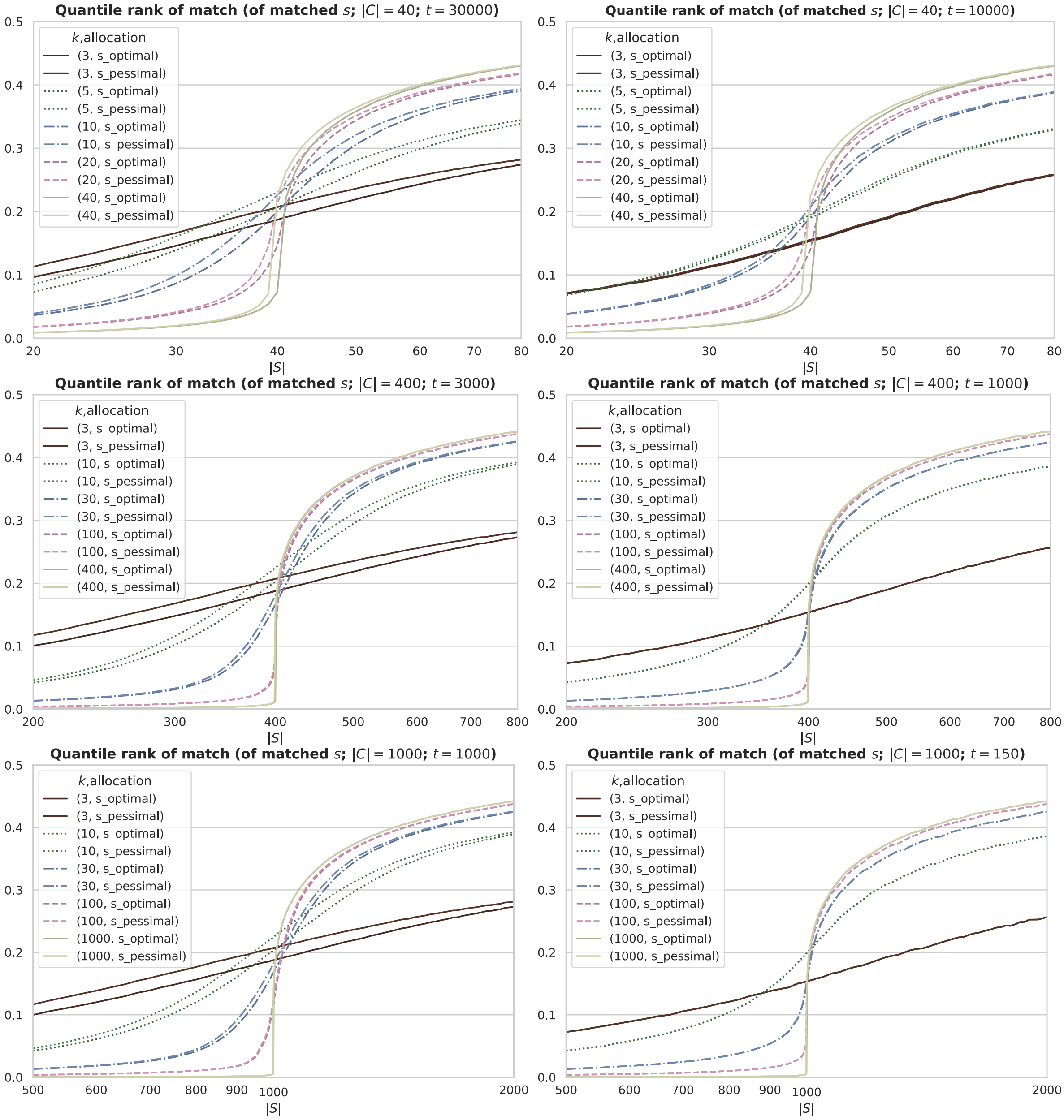

Figure 10: Matched students' quantile rank of matches under k-nearest-colleges preferences (left) and similarly-selective homogeneous preferences (right). Plots at left reproduce parts of Figures 1, 3, and 6.

D A generalized locality property

The proof of Theorem 3.1 above demonstrates the existence of non-vanishing market power in any large market model in which a variant of Lemma 3.2 obtains, namely where $\lim_{n\to\infty} E_{c_1}^{(n)} > 0$. In this appendix, I characterize a spectral condition on preference structure, with a natural interpretation as a measure of locality, that is sufficient to demonstrate that a similar bound obtains. In § D.2 below, I demonstrate that many random market models analogous to models common in the network-structure literature satisfy the network-structural analogue of this condition, and therefore analogous preference structures generate markets which exhibit non-vanishing market power.

D.1 Uniform students

Let $\tilde{\Gamma}^{(n)} = \left(C^{(n)}, S^{(n)}, \mathcal{P}_C^{(n)}, \mathcal{P}_S^{(n)}\right)$ be a random market with $n = \left|C^{(n)}\right|$ and $\mathcal{P}_C^{(n)}$ the uniform distribution over permutations of $S^{(n)}$. Fix some c_1 and $c_2 \in C^{(n)}$, and draw a s_1 and s_2 with preferences from $\mathcal{P}_S^{(n)}$. Without loss of generality, let $s_1 \succ_{c_1} s_2$. Consider then the following events:

- $E_{[c_1;c_2]}^{(n)} := s_2 \succ_{c_2} s_1$
- $E_{[c_1;\forall s]}^{(n)} := \forall s \in S^{(n)} \backslash \{s_1, s_2\}, (\varnothing \succ_s c_1) \wedge (\varnothing \succ_s c_2)$
- $E_{[c_1;s_1]}^{(n)} := (\mathrm{r}_{s_1}(c_2) = 1) \wedge (\mathrm{r}_{s_1}(c_1) = 2)$
- $E_{[c_1;s_2]}^{(n)} := (\mathrm{r}_{s_2}(c_1) = 1) \wedge (\mathrm{r}_{s_2}(c_2) = 2)$
- $E^{(n)}(s_1, s_2, c_1, c_2) = E_{[c_1;c_2]}^{(n)} \wedge E_{[c_1;\forall s]}^{(n)} \wedge E_{[c_1;s_1]}^{n} \wedge E_{[c_1;s_2]}^{(n)}$
- $E_{c_1}^{(n)} := \bigcup_{(s_1,s_2,c_2)\in S^{(n)}\times S^{(n)}\times C^{(n)}} E^{(n)}(s_1, s_2, c_1, c_2)$.

Since $\mathcal{P}_C^{(n)}$ is uniform over permutations of students, $P\left[E_{[c_1;c_2]}^{(n)}\right] = \frac{1}{2}$. And since students' preferences are drawn independently, consider the remaining events as independent probabilities on draws of a student s from $\mathcal{P}_S^{(n)}$:

- $P\left[E_{[c_1;\forall s]}^{(n)}\right] = \left(P_s[\varnothing \succ_s c_1] \cdot P_s[\varnothing \succ_s c_2 \mid \varnothing \succ_s c_1]\right)^{\left(\left|S^{(n)}\right|-2\right)}$
- $P\left[E_{[c_1;s_1]}^{(n)}\right] = P_s[\mathrm{r}_s(c_2) = 1] \cdot P_s[\mathrm{r}_s(c_1) = 2 \mid \mathrm{r}_s(c_2) = 1]$

- $P\left[E^{(n)}_{[c_1;s_2]}\right] = P_s[\mathrm{r}_s(c_1) = 1] \cdot P_s[\mathrm{r}_s(c_2) = 2 \mid \mathrm{r}_s(c_1) = 1]$.

So, noting that for different selections of (s_1, s_2, c_2) the events $E^{(n)}(s_1, s_2, c_1, c_2)$ are disjoint and that there are $\frac{n(n-1)}{2}$ draws of (s_1, s_2) such that $s_1 \neq s_2$ and $s_1 \succ_{c_1} s_2$, see:

$$
\begin{aligned}
P[E^{(n)}_{c_1}] =& \frac{n(n-1)}{4} \sum_{c_2 \in C^{(n)} \setminus \{c_2\}} P\left[E^{(n)}_{[c_1;\forall s]}\right] \cdot P\left[E^{(n)}_{[c_1;s_1]}\right] \cdot P\left[E^{(n)}_{[c_1;s_2]}\right] \\
=& \frac{n(n-1)}{4} \cdot P_s[\mathrm{r}_s(c_1) = 1] \cdot P_s[\varnothing \succ_s c_1]^{\left(|S^{(n)}|-2\right)} \\
& \cdot \sum_{c_2 \in C^{(n)} \setminus \{c_1\}} \Big(P_s[\mathrm{r}_s(c_2) = 1] \\
& \qquad \cdot P_s[\varnothing \succ_s c_2 \mid \varnothing \succ_s c_1]^{\left(|S^{(n)}|-2\right)} \\
& \qquad \cdot P_s[\mathrm{r}_s(c_1) = 2 \mid \mathrm{r}_s(c_2) = 1] \\
& \qquad \cdot P_s[\mathrm{r}_s(c_2) = 2 \mid \mathrm{r}_s(c_1) = 1] \Big).
\end{aligned}
\tag{D.1}
$$

We can use this expression to derive from preference structure a bound on the expected number of colleges (and students) with more than one stable match.

D.1.1 Uniform students, *ex ante* equipopular colleges

Say that colleges are *ex ante equipopular*[23] if

$$
\forall i \in \{0, \ldots, k\}, \forall c \in C, P_s[\mathrm{r}_s(c) = i] = \frac{1}{|C|},
\tag{D.2}
$$

i. e., if each college is equally likely to appear in each position in a student's preference list.

Let $\left(\tilde{\Gamma}^{(n)}\right)_{n \in \mathbb{N}}$ be a $(k, \overline{q})$-regular sequence of random markets with colleges *ex ante* equipopular and $\mathcal{P}^{(n)}_C$ the uniform distribution over permutations of $S^{(n)}$. Consider $\tilde{\Gamma}^{(n)} = \left(C^{(n)}, S^{(n)}, \mathcal{P}^{(n)}_C, \mathcal{P}^{(n)}_S\right)$. Then note that

[23]Note that *ex ante* equipopularity allows for correlations within a students' rankings of different colleges, and is a weaker condition than *ex ante* symmetry (as it allows for asymmetries in correlation structure).

$P_s[\varnothing \succ_s c_2 \mid \varnothing \succ_s c_1] \geq \frac{n-2k}{n}$ and so:

$$\begin{aligned} P[E_{c_1}^{(n)}] \geq & \frac{n(n-1)}{4} \cdot \frac{1}{n} \cdot \left(1 - \frac{k}{n}\right)^{(\overline{q}n-2)} \\ & \cdot \sum_{c_2 \in C^{(n)}} \Bigg(\frac{1}{n} \cdot \left(1 - \frac{2k}{n}\right)^{(\overline{q}n-2)} \\ & \qquad \cdot P_s[\mathrm{r}_s(c_1) = 2 \mid \mathrm{r}_s(c_2) = 1] \\ & \qquad \cdot P_s[\mathrm{r}_s(c_2) = 2 \mid \mathrm{r}_s(c_1) = 1] \Bigg), \end{aligned} \tag{D.3}$$

which in the $n \to \infty$ limit is:

$$\begin{aligned} \lim_{n\to\infty} P[E_{c_1}^{(n)}] \geq \frac{\exp[-3\overline{q}k]}{4} \cdot \sum_{c_2 \in C^{(n)}} \Bigg(& P_s[\mathrm{r}_s(c_1) = 2 \mid \mathrm{r}_s(c_2) = 1] \\ & \cdot P_s[\mathrm{r}_s(c_2) = 2 \mid \mathrm{r}_s(c_1) = 1] \Bigg). \end{aligned} \tag{D.4}$$

Define the stochastic operator $A^{(n)} := \big[P_s[\mathrm{r}_s(c_i) = 2 \mid \mathrm{r}_s(c_j) = 1]\big]_{ij}$. Then note:

$$\begin{aligned} \Big(\big(A^{(n)}\big)^T\big(A^{(n)}\big)\Big)_{c_1c_1} = \sum_{c_2 \in C^{(n)}} \Bigg(& P_s[\mathrm{r}_s(c_1) = 2 \mid \mathrm{r}_s(c_2) = 1] \\ & \cdot P_s[\mathrm{r}_s(c_2) = 2 \mid \mathrm{r}_s(c_1) = 1] \Bigg) \end{aligned} \tag{D.5}$$

and

$$\lim_{n\to\infty} \mathbb{E}\big[c \in C^{(n)} : E_c^{(n)}\big] \geq \frac{\exp[-3\overline{q}k]}{4} \lim_{n\to\infty} \mathrm{Tr}\Big(\big(A^{(n)}\big)^T\big(A^{(n)}\big)\Big). \tag{D.6}$$

That is, we can express a sufficient condition that a market exhibits non-vanishing market power (under the above regularity and uniformity assumptions) as a purely spectral condition on the matrix of first- and second-choice colleges by students.

As an example, we can demonstrate non-vanishing market power in the setup of Theorem 3.1[24] by noting *ex ante* equipopularity, regularity, uniformity of colleges' preferences, and that $\mathrm{Tr}\big(\big(A^{(n)}\big)^T\big(A^{(n)}\big)\big) = \frac{n}{k}$, implying that the expected fraction of colleges with more than one stable match is non-vanishing.

[24]albeit with a weaker bound in terms of k and $\overline{q}$

D.2 Locality in network structure

Modeling locality in inter-agent networks was an early motivation for random graph models that generalized uniform (Gilbert (1959); Erdős and Rényi (1960)) and homogeneous power-law (Barabási and Albert (1999); Albert and Barabási (2000)) network models. For the results in the main text, I considered an analogue of a synthetic locality model presented by Watts and Strogatz (1992), which mixed an Erdős–Rényi uniform graph model with a regular ring lattice. In the network structure literature, locality was also introduced to generative models via variation to their sequential-attachment mechanisms (Kleinberg *et al.* (1999); Kumar *et al.* (2000); Leskovec *et al.* (2005)). Further work has refined the modeling of locality structure in both synthetic (Leskovec *et al.* (2010)) and ad-hoc (Yang and Leskovec (2014)) network models.

The locality condition established above is a form of subgraph density condition on the ordered, directed graph of agent preferences, similar to the triangle-counting interpretation of network locality in undirected graphs (Wasserman and Faust (1994)). As such, a number of proposed models from the network-structure literature have preference-structure analogues that exhibit non-vanishing locality, in the above sense:

- ring-lattice mixture models (Watts and Strogatz (1992)), as introduced in the main text,
- geography-based models (Jin *et al.* (2000)),
- copying-based (Kleinberg *et al.* (1999); Kumar *et al.* (2000)) and prototype-copying-based (Leskovec *et al.* (2005)) preferential attachment models,
- Kronecker-product models (Chakrabarti *et al.* (2004); Leskovec *et al.* (2010)),
- community-attribute models (Yang and Leskovec (2014)).

I anticipate scope for future work to investigate the applicability of network models to modeling preference structure in matching markets—and the matching-market dynamics that they imply.

E A tighter bound on multiple-stable-allocations event probability

Consider a regular sequence of balanced, one-to-one, k-nearest-colleges random markets.

Lemma E.1. *Fix an arbitrary $\epsilon > 0$. There exists sufficiently large n such that for each college $c_1 \in C^{(n)}$ the event*

$$E^{(n)}_{c_1} := \bigcup_{(s_1,s_2,c_2)\in S^{(n)}\times S^{(n)}\times C^{(n)}} E^{(n)}(s_1,s_2,c_1,c_2) \tag{E.1}$$

has probability bounded below by $\frac{\exp[-k-1]}{4k^2}\cdot\frac{\gamma(k-\gamma)^2}{(k-1)^2}-\epsilon$, where $\gamma := \frac{e}{e-1} \approx 1.58$.

Proof. Fix an arbitrary $c_1 \in C^{(n)}$. For different selections of (s_1, s_2, c_2), the events $E^{(n)}(s_1, s_2, c_1, c_2)$ are disjoint. There are $n \cdot (n-1) \cdot (k-1)$ possible selections of (s_1, s_2, c_2) such that $s_1 \neq s_2$ and $1 \leq y_{c_2} - y_{c_1} < k$. Half of these events have zero probability (when $s_2 \succ_{c_1} s_1$). Letting $\ell := y_{c_2} - y_{c_1}$, the probability of each of the other events is at least

$$\frac{1}{2}\cdot\left(1-\frac{k+\ell}{n-1}\right)^{(n-2)}\cdot\frac{k-\ell}{nk(k-1)}\cdot\frac{k-\ell}{nk(k-1)}, \tag{E.2}$$

where each term in this expression corresponds to an (independent)[25] requirement from the definition of $E^{(n)}(s_1, s_2, c_1, c_2)$. Thus, the probability of the event $E^{(n)}_{c_1}$ is at least

$$n\cdot(n-1)\cdot\sum_{\ell=1}^{k-1}\left(\frac{1}{2}\cdot\frac{1}{2}\cdot\left(1-\frac{k+\ell}{n-1}\right)^{(n-2)}\cdot\frac{k-\ell}{nk(k-1)}\cdot\frac{k-\ell}{nk(k-1)}\right), \tag{E.3}$$

and the limit inferior of this expression as $n \to \infty$ is, letting $\gamma := \frac{e}{e-1} \approx 1.58$,

$$\frac{\exp[-k]}{4k^2(k-1)^2}\left(\sum_{\ell=1}^{k-1}\exp[-\ell]k^2 - 2\sum_{\ell=1}^{k-1}\exp[-\ell]k\ell + \sum_{\ell=1}^{k-1}\exp[-\ell]\ell^2\right). \tag{E.4}$$

With the following explicit forms for the summations:

$$\sum_{\ell=1}^{k-1}\exp[-\ell]k^2 = \exp[-1]k^2\gamma - \exp[-k]k^2\gamma \tag{E.5}$$

[25] To be precise, the second, third, fourth, and fifth requirements, respectively.

$$\begin{aligned}
&\sum_{\ell=1}^{k-1} \exp[-\ell] k\ell \\
&= k \sum_{i=1}^{k-1} \sum_{\ell=i}^{k-1} \exp[-\ell] \\
&= k \sum_{i=1}^{k-1} (\exp[-i] - \exp[-k])\gamma \\
&= k\gamma \left(-\exp[-k](k-1) + \sum_{i=1}^{k-1} \exp[-i] \right) \\
&= k\gamma(-\exp[-k](k-1) + (\exp[-1] - \exp[-k])\gamma) \\
&= \exp[-1]k\gamma^2 - \exp[-k]\big(k(k-1)\gamma + k\gamma^2\big)
\end{aligned} \tag{E.6}$$

$$\begin{aligned}
&\sum_{\ell=1}^{k-1} \exp[-\ell]\ell^2 \\
&= \sum_{i=1}^{k-1} \sum_{j=i}^{k-1} \sum_{\ell=j}^{k-1} \exp[-\ell] \\
&= \sum_{i=1}^{k-1} \sum_{j=i}^{k-1} (\exp[-j] - \exp[-k])\gamma \\
&= \gamma \left(-\exp[-k] \frac{k(k-1)}{2} + \sum_{i=1}^{k-1} \sum_{j=i}^{k-1} \exp[-j] \right) \\
&= \gamma \left(-\exp[-k] \frac{k(k-1)}{2} + \sum_{i=1}^{k-1} (\exp[-i]\gamma - \exp[-k]\gamma) \right) \\
&= \gamma \left(-\exp[-k] \frac{k(k-1)}{2} - \exp[-k]\gamma(k-1) + \gamma \sum_{i=1}^{k-1} \exp[-i] \right) \\
&= \gamma \left(-\exp[-k] \frac{k(k-1)}{2} - \exp[-k]\gamma(k-1) + \exp[-1]\gamma^2 - \exp[-k]\gamma^2 \right) \\
&= \exp[-1]\gamma^3 - \exp[-k] \left(\frac{k(k-1)}{2}\gamma + (k-1)\gamma^2 + \gamma^3 \right),
\end{aligned} \tag{E.7}$$

we recover an explicit form for the limit:

$$
\begin{aligned}
&\lim_{n\to\infty}\left[n\cdot(n-1)\cdot\sum_{\ell=1}^{k-1}\left(\frac{1}{2}\cdot\frac{1}{2}\cdot\left(1-\frac{k+\ell}{n-1}\right)^{2n}\cdot\frac{k-\ell}{nk(k-1)}\cdot\frac{k-\ell}{nk(k-1)}\right)\right]\\
&=\frac{\exp[-k]}{4k^2(k-1)^2}\left(\sum_{\ell=1}^{k-1}\exp[-\ell]k^2-2\sum_{\ell=1}^{k-1}\exp[-\ell]k\ell+\sum_{\ell=1}^{k-1}\exp[-\ell]\ell^2\right)\\
&=\frac{\exp[-k]}{4k^2(k-1)^2}\Bigg(\exp[-1]\big(k^2\gamma-2k\gamma^2+\gamma^3\big)\\
&\qquad\qquad+\exp[-k]\left(\left(\frac{k^2-3k}{2}\right)\gamma+(k+1)\gamma^2-\gamma^3\right)\Bigg)\\
&>\frac{\exp[-k]}{4k^2(k-1)^2}\cdot\exp[-1]\big(k^2\gamma-2k\gamma^2+\gamma^3\big)\\
&=\frac{\exp[-k-1](k-\gamma)^2\gamma}{4k^2(k-1)^2}.
\end{aligned}
\tag{E.8}
$$

For $k \geq 3$, this is greater than the bound of $\frac{\exp[-k-1]}{4k^2}$ derived in lemma 3.2 by considering only interactions between adjacent colleges. In the $k \to \infty$ limit, the tighter bound is tighter by a factor of $\gamma \approx 1.58$.

□