Large random matching markets with localized preference structures can exhibit large cores*

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Abstract

I present a class of models for random matching markets with non-homogeneous agent preferences, drawn from the computer science literature on network structure. An analogue of the Watts–Strogatz (1998) 'small-world' network model supports significant incentives to manipulate matching outcomes. The scope for manipulation remains substantial as markets become large and unbalanced—contrasting prior work which found little scope under uniform or homogeneous random preferences. This scope for manipulation directly corresponds to core size and differences in agents' welfare between core outcomes. These results suggest largeness and cross-side imbalance may be insufficient to fully explain small cores in matching markets; I discuss alternative explanations.

Keywords: Matching; Large markets; Incentives

1 Introduction

I consider matching markets where both sides have preferences over potential matches, and no contract details allow for transfer of utility. Examples of importance and interest include allocation of students to schools at primary

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through graduate levels, allocation of migrants to settlement destinations, matching of participants in bipartite dating pools, construction of exclusive social organizations, and some settings of assignment of entry-level workers to employer firms.

As matching markets are used to coordinate and allocate the labor, training, and education of hundreds of thousands of participants each year—typically in multi-year commitments—their efficiency can be of substantial economic importance. Correspondingly, the outcomes of matching can be of such importance to participants that markets provide significant incentives for strategic manipulation by sophisticated participants (Pathak and Sönmez (2008)). Thus, matching-market designers are directly concerned with the choices and incentives that mechanisms offer to market participants, and their effects on allocation outcomes in equilibrium (Roth (2002)).

Previous theoretical studies of matching markets have suggested that agents' abilities and incentives to manipulate markets generally vanish as the size of a random market grows (Roth and Peranson (1999); Immorlica and Mahdian (2005)[1]). I find, however, that these theoretical results depend crucially on the agent preferences lacking *locality structure*; in matching markets with localized preference structures, agents' incentives to manipulate can instead robustly fail to vanish. These findings—that the received wisdom that cores robustly vanish in large markets depends on assumptions about agent preference structure—further contextualize results presented by Hassidim *et al.* (2018) and Biró *et al.* (2022) who demonstrate non-vanishing cores in a random model of matching with contracts.

The results presented in the present work suggest that, in large markets where agents' preferences will plausibly exhibit nontrivial locality, empirical observations of small cores may not be adequately explained by market size alone. They are also not adequately explained by competition effects, as the non-vanishing cores are robust to imbalance between the sides of the market—an outcome not observed in models of markets without preference locality.[2] While some forms of correlation between agents' preferences pro-

[1] As this paper was in revision, Immorlica, Mahdian, and the authors of Ashlagi, Kanoria, and Leshno (2013) were recognized by the ACM Special Interest Group on Economics and Compution with the 2023 "Test of Time" award for their respective papers. I congratulate the recipients on this well-deserved award. The award citation "for explaining an apparent gap between the theory and practice of matching markets and helping us understand why small cores are so common" highlights enduring interest in the effects which cause cores to be large or small in matching markets, in which spirit the present work is presented.

[2] A possible interpretation of this interaction is that in order for a small amount of competition to cause a global lack of market power, a large number of agents across the

vide alternate explanations for empirically small cores, the present work suggests that further study of the interaction of size, competition, and preference structure in markets of interest is needed to produce a full explanation of these observations and an understanding of potential settings where small cores may not be guaranteed.

1.1 Motivation and background

The literature on matching-market mechanism design has shown that *ex post stability* of outcomes, and *ex ante incentive-compatibility* of mechanisms, are of primary importance to the success of an allocation marketplace (Roth (1984, 2002)). Intuitively speaking, *ex post* stability is the condition that it is an equilibrium for agents to follow the assigned allocation, rather than seeking a match outside it or electing to later recontract. Gale and Shapley (1962) have shown that, in one-to-one[3] matching markets, stable allocations exist in general and are efficiently computable. In fact, stable-allocation-finding mechanisms are now used in a variety of settings where 'unraveling' deviation from mechanism assignments would otherwise pose significant welfare costs to the market (Roth (1984); Roth and Xing (1994); Niederle and Roth (2003); Fréchette *et al.* (2007)).

However, stability does not necessarily determine a unique allocation (Knuth (1976)). Multiple stable allocations may exist, and individual agents may have sufficient market power to influence which equilibrium obtains. More explicitly, a nontrivial *core* (which, in this setting, coincides with the set of *ex post* stable allocations) excludes the possibility of a core-selecting mechanism for which truth-telling is a dominant strategy for all participants. An agent who selects a desirable allocation from the true core—and (mis)represents that only that allocation's outcome is acceptable to them—will be granted that outcome by a core-selecting mechanism (Gale and Sotomayor (1985)). Thus, agents deviating from truth-telling can manipulate a core-selecting mechanism into assigning them their most-favored outcome among those achieved in the core.

Therefore, *ex ante* incentive-compatibility[4] of a core-selecting mechanism

market need to be essentially substitutes for one another. Analogous effects are well-known in price theory. However, the present work suggests that localized preferences can cause agents to be effectively differentiated and retain local market power. I thank an anonymous referee for the suggestion of this interpretation.

[3]In many-to-one and many-to-many settings, existence of stable allocations generally requires either full substitutability (Roth and Sotomayor (1990)) or full complementarity (Rostek and Yoder (2020)).

[4]*i. e.*, compatibility of a truth-telling strategy with agents' own incentives.

can only be assured for agents who are assigned their most-favored core outcome when truth-telling. Roth (1982) demonstrates that this can, at best, be simultaneously assured for all agents on one side of the market (unless only one core allocation exists).

It follows that the available incentives for strategic manipulation are closely linked to the gap in welfare (of the mechanism-disfavored side) between the subjectively optimal and pessimal core outcomes.[5] Strategic manipulation degrades welfare by risking allocative inefficiency (and creating its own wasteful investment race) and threatens primary objectives of mechanism design. Agents' welfare gaps among core allocations are therefore of direct concern to the matching-market designer.

Prior work has investigated intra-core welfare gaps by bounding core size. Roth and Peranson (1999) observed qualitatively small cores in the National Resident Matching Program and they conjectured, from simulation results, that fixed-size preference lists drawn uniformly at random yielded vanishingly small cores as markets became large. Immorlica and Mahdian (2005) proved this conjecture. Kojima and Pathak (2009) and Ashlagi *et al.* (2017) demonstrated comparable results for many-to-one markets with short preference lists (whether balanced or unbalanced) and unbalanced one-to-one markets with long preference lists, respectively.

This literature has thus presented several classes of random market models under which core size and welfare gap size vanish as the number of agents in the market becomes large. However, these models have nearly all been *homogeneous*—that is, each agent will form each preference with probability functionally independent of the other preferences they form.

The present work constrasts these results by presenting a class of random market models made non-homogenous by a *locality* structure in agent preferences. These models are inspired by analogy to a class of models for inter-agent networks drawn from the computer science literature on network structure, which were originally presented to model empirical observations about network formation.[6] In these models with locality, agents have latent features that place them at a point in feature space, and some agents prefer to match to agents 'nearby' in space. This structure can reflect not just geographic structure in agents' preferences, but also structure in preferences

[5]Furthermore, the presence of manipulation by the mechanism-disfavored side of the market creates game-theoretic incentives for the mechanism-*favored* side of the market to manipulate in order to discourage the other side's manipulation. Gale and Sotomayor (1985) analyze the strategic equilibrium of this game in the perfect-information setting.

[6]I further discuss the relevant network-structure literature in § 1.4 below; for another treatment, see Chapter 2 of Rheingans-Yoo (2016).

based on categorical or stylistic features of potential matches.[7]

In the present work, I demonstrate that, under random market models where agents' preferences exhibit sufficient locality, optimal–pessimal welfare gaps remain substantial as markets become large, as does the fraction of agents facing such gaps.

1.2 Overview of results

My model of college admissions markets can be understood as extending the Kojima and Pathak (2009) model of large matching markets. Their model is characterized by (a) a large number of colleges, of which each student finds only a small number acceptable, and (b) random, conditionally independent agent preferences. I weaken the latter condition and allow a student's opinion of one college to correlate with their opinion of colleges nearby in latent feature space. I maintain *ex ante* symmetry between schools, and between students, for my main results. When student preferences exhibit this form of locality, I find that the core is non-vanishing as the market becomes large.

I additionally investigate the effect of *cross-side imbalance* on markets with locality. Ashlagi *et al.* (2017) find that when agents have long preference lists, balanced markets exhibit non-vanishing cores but introducing a small cross-side imbalance causes core size to vanish. However, I find no such effect in random markets with my model of preference locality.

I augment these theoretical results with simulation results that demonstrate that core size and manipulation incentives are substantial in magnitude in practice. These simulations demonstrate the persistence of a large core in large markets and under cross-side imbalance, and show that the otherwise sharp welfare advantage afforded to the short side of a homogeneous market (Kanoria *et al.* (2021)) can be attenuated by preference locality. Finally, I discuss plausible alternative explanations for empirical observations of small cores in some large matching markets with preference locality.

1.3 Related work on non-homogeneous markets

The two examples of robustly[8] non-vanishing market power in the literature are are in replica markets and the Biró *et al.* (2022) setting of college admissions with financial aid contracts. Other non-homogeneous random market

[7] For discussion of empirical observations of locality structure in the National Resident Matching Program, see § 5.1 below.

[8] Ashlagi *et al.* (2017) show that non-vanishing core size and market power is only supported by long preference lists in the knife-edge case of cross-side balance.

models presented for study have not exhibited large cores, though they have either not included locality, or have combined locality with other structural properties that are understood to cause core convergence.

1.3.1 Random replica markets

A *random replica market* is the union of of k mutually disconnected random submarkets. In this form, a large replica market can have a large total number of agents n, but with each agent only interacting with a submarket of a small fixed size n/k. It is commonly understood that large replica markets need not exhibit vanishing core size if k grows as $O(n)$ and submarket size remains $O(1)$.

For example, Immorlica and Mahdian (2015), in their Remark 2.7, consider a random replica market of non-interacting 2-college submarkets to which students are assigned at random. In this model they find that the fraction of agents with more than one stable match fails to vanish as the market grows large in the number of submarkets and with proportionally many students. This observation, however, fails to illuminate matching behaviors within the giant connected component of a matching market, which virtually always dominates real-world markets of interest.

By contrast, the Watts–Strogatz model of the present work's main results allows random markets to be fully connected, with average agent-to-agent distance of order $O(\log(n))$, and demonstrates that non-vanishing market power is possible in such settings. It is not commonly known that markets this thoroughly connected can exhibit large cores.

1.3.2 Large cores in matching with contracts and the role of preference heterogeneity

Biró *et al.* (2022) present a setting of matching with contracts where market power does not vanish in the large-market limit. The proofs presented in the present work use techniques for constructively demonstrating market power in the large-market limit similar to theirs, but adapted to the setting of matching without contracts and random market models with locality in preference structure.

One difference required by the setting of matching without contracts, however, is the need to impose heterogeneity in preferences on *both* sides of the market. It is well-known that when one or both sides of a matching market have preferences that align with a single common ranking, only one stable matching exists.[9] This is no obstacle to small-cores results in the literature on matching without contracts, but means that results demonstrating the potential for large cores will necessarily be restricted to settings where both sides' preferences exhibit at least some heterogeneity. For my main result, I assign colleges uniform random preferences to achieve this condition.

Biró *et al.* (2022) are able to avoid this heterogeneity requirement for their results because their setting is matching with contracts, where multiple stable matchings can be supported within a single student–college pair. Indeed, their Proposition 5 indicates that when students' preferences over matches both align and dominate their preferences over contract terms, the only multiplicities in the set of stable allocations come from varying contract terms between matched partners.

1.3.3 Other work on non-homogeneous markets

Kojima and Pathak (2009) consider the union of a bounded number of homogeneous models, to represent agent types. However, with a bounded number of agent types, this model exhibits vanishing locality, as discussed in Appendix C.

Ashlagi *et al.* (2017) consider a multiparameter model of correlated preference structure, including a parameter for locality, in their setting with long preference lists. However, their locality parameter simultaneously induces alignment between sides of the market, shrinking the possible difference between the student-optimal and college-optimal stable matches. Unsurprisingly, they find that increasing this parameter does little to increase the size of the core (though it does cause a slight increase under sufficiently unbalanced market specifications). By contrast, I present a model of preference locality *without* cross-side preference alignment, and find large cores supported under certain market specifications.

1.4 Related work on network structure

The approach to modeling preference structure presented in the present work draws from the computer science literature on network structure in large

[9] Any stable matching must coincide with the unique matching produced by serial dictatorship by the commonly-ranked agents, in order of their rank ($\mathrm{SD}_{\mathrm{rank}}$). Any matching μ which does not coincide is blocked by the pair consisting of x^*, the top-ranked agent on the commonly-ranked side *not* receiving their $\mathrm{SD}_{\mathrm{rank}}$ match (who thus receives a match they rank lower); and x^*'s $\mathrm{SD}_{\mathrm{rank}}$ match, who receives a match in μ that they rank lower than x^*.

graphs. Early analysis of large homogeneous graphs (Gilbert (1959); Erdős and Rényi (1960)) was conducted via combinatorial techniques similar to the existing literature on large matching markets. However, the contemporary network-structure literature has turned to non-homogeneous models to model structural features empirically observed in real-world graphs—*e. g.*, locality over latent features (Watts and Strogatz (1998); Leskovec *et al.* (2009)); the small-world property (Leskovec *et al.* (2005); Bordino *et al.* (2008)); power-law distribution of node degree (Barabási and Albert (1999); Aiello *et al.* (2000)); self-similar subnetwork structure (Chakrabarti *et al.* (2004)); and others (Boldi *et al.* (2011); Yang and Leskovec (2014)).

The specific preference-locality structure I introduce is inspired in form by one-dimensional locality models for network graphs (Watts and Strogatz (1998)). While I do not consider the analogues of more-sophisticated generative network models in this work, those models do suggest that locality can exist under a wide and general variety of random market models, and in a form that supports market power in the large-market limit. In Appendix C, I present a generalized locality property sufficient to yield my main results and suggest common generative network models from the network-structure literature under which the analogous property obtains.

Market agents forming preferences over potential matches are plausibly influenced by similar processes as agents forming ties in a network graph. On this basis, I propose that work on the theory of matching should be able to fruitfully build on the network-structure literature to better model the structures of matching markets among agents embedded in the actual world, and explore the welfare implications and strategic incentives that arise from them.

2 Model

I extend the random matching market model of Kojima and Pathak (2009) to support non-homogeneous structure among students' preferences. This model captures "correlated preferences" models of the kind discussed and simulated in Ashlagi *et al.* (2017), allowing for direct comparison of homogeneous and non-homogeneous preference structures.

2.1 Matching markets

I consider one-to-one and many-to-one matching markets, with *colleges* demanding $q \geq 1$ matches and *students* demanding precisely one match.

Given a set of colleges C and a set of students S (together *agents*), each college c has a complete strict preference relation $\succ_c$ over the subsets of students 2^S and each student s has a complete strict preference relation $\succ_s$ over the colleges C and the outcome of being unmatched (denoted $\varnothing$). A student s is *acceptable* to a college c if $\{s\} \succ_c \varnothing$ and a college c is *acceptable* to a student s if $c \succ_s \varnothing$. A *matching market* is a tuple of colleges, students, and agent preferences.

For a college c and a *quota* q_c, the preference relation $\succ_c$ is *responsive with quota* q_c if the ranking of a student is independent of their colleagues, and all sets of students exceeding quota q_c are unacceptable (see Roth (1985) for further discussion). I consider only responsive college preferences in the present work.[10] Furthermore, I abuse notation and write $s \succ_c \varnothing$ and $s_1 \succ_c s_2$ to indicate $\{s\} \succ_c \varnothing$ and $\{s_1\} \succ_c \{s_2\}$ when discussing colleges' preferences with respect to individual students.

A *matching* is a mapping μ on $C \cup S$ that associates colleges to disjoint sets of students, and students to the corresponding college or unmatched outcome:

- For $c \in C$, $\mu(c) \in 2^S$.
- For $s \in S$, $\mu(s) \in C \cup \varnothing$.
- For $c, s \in C \times S$, $s \in \mu(c) \iff \mu(s) = c$.

A matching is *feasible* if each college c is matched to a number of students no more than q_c, and is *individually rational* if:

- Each college is only matched to acceptable students.
- Each student is matched to an acceptable college or the unmatched outcome.

A matching μ is *blocked* by a college–student pair c, s if:

- s prefers c to their match ($c \succ_s \mu(s)$).
- Either c has a vacancy and finds s acceptable ($|\mu(c)| < q_c$ and $s \succ_c \varnothing$), or c prefers s to some other matched student ($\exists s' \in \mu(c) : s \succ_c s'$).

A matching is *stable* if it is feasible, individually rational, and unblocked.

[10] Kojima and Pathak (2009) note that every responsive preference relation corresponds to an additive utility function over students.

2.2 Core size and market power

In matching markets, the *core*[11] coincides with the set of stable matchings (Roth (1985)). Following Immorlica and Mahdian (2005), I consider *core size* in terms of the fraction of agents with multiple stable matches[12] and formally describe the core as *small* or *vanishing* if the expected fraction of agents with multiple stable matches asymptotically vanishes as a market becomes large in the number of agents. Correspondingly, the core is formally *large* or *non-vanishing* if the expected fraction fails to vanish as the market becomes large.

In matching markets, core size creates scope for agents to manipulate core-selecting mechanisms (Gale and Sotomayor (1985)). Kojima and Pathak (2009) argue that manipulability is best understood in terms of *market power*—the ability for an agent's strategic rejection of a proposal to affect the set of other proposals that agent will later observe—and I adopt this term in the same sense where appropriate.

2.3 Random markets

Given a set of colleges C and a set of students S, a *random market* is a tuple $\tilde{\Gamma} = \big(C, S, \mathcal{P}_C, q, \mathcal{P}_S\big)$, where $\mathcal{P}_C$ is a probability distribution on orderings of $S \cup \{\varnothing\}$, $q = (q_c)_{c \in C}$ is a vector of college quotas, and $\mathcal{P}_S$ is a probability distribution on orderings of $C \cup \{\varnothing\}$. Each random market induces a market by randomly generating preferences of each college c by drawing from $\mathcal{P}_C$, giving each college c quota q_c, and generating preferences of each student s by drawing from $\mathcal{P}_S$.[13]

2.3.1 Kojima–Pathak random markets

Kojima and Pathak (2009) present a special case of this random market model for $\mathcal{P}_C$ a fixed realization of complete colleges' preferences and $\mathcal{P}_S$ given as follows:

- Fix $k > 0$ a positive integer.

[11] *i. e.*, the set of outcomes on which no coalition can unilaterally improve.

[12] *i. e.*, matches achieved in some stable matching.

[13] As in the referenced works, I consider the possibility of manipulations under complete information; randomness over preferences is introduced only to assess the frequency of situations in which agents have incentives to manipulate. An expectation over colleges' preferences $\mathcal{P}_C$ is necessary to demonstrate large cores without contracts, unlike in the referenced works, as discussed in § 1.3.2.

- Fix $\mathcal{D} = (p_c)_{c \in C}$ a probability distribution on C.
- Assign each student's preferences by drawing k colleges from $\mathcal{D}$ without replacement, then appending $\varnothing$ (whereafter the order of successive colleges is immaterial).

Effectively, each students' preference-ordering of colleges is composed of k independent draws from a common distribution on colleges $\mathcal{D}$. I hereafter call random markets of this form *Kojima–Pathak random markets.*

2.3.2 Watts–Strogatz random markets

Next, I consider a simple linear model of preference locality structure, closely analogous to the model proposed by Watts and Strogatz (1998) to model locality in network structure. For a positive integer k, let a *uniform, 1-dimensional Watts–Strogatz random market* (hereafter *Watts–Strogatz random market* when not otherwise qualified) with radius k and locality parameter ℓ be a random market $\tilde{\Gamma} = \left(C, S, \mathcal{P}_C, q, \mathcal{P}_S\right)$ with $\mathcal{P}_C$ the uniform distribution over complete permutations of students and $\mathcal{P}_S$ given as follows:

- Arrange the colleges C uniformly on a circle.
- Place each student s uniformly at random at a point on the circle.
- Let each student pick ℓ colleges from the nearest k colleges, and $k - \ell$ other colleges from C uniformly, with all other colleges unacceptable.[14]
- Each student draws preferences uniformly over the colleges they find acceptable.

Note that k controls the length of preference lists as well as the radius of the locality neighborhood.[15] ℓ/k can be considered a measure of the intensity of locality; $\ell = k$ corresponds to *full locality*, where students choose only from a defined neighborhood, and $\ell = 0$ corresponds to the fully nonlocal Kojima–Pathak model. Hereafter, I will require $\ell \geq 2$ in discussion of Watts–Strogatz random markets, as $\ell = 0$ and $\ell = 1$ are degenerate cases with no locality structure.

In Appendix C, I further generalize the relevant conditions on preference locality and provide more general models from the network-structure literature with preference-structure analogues that support large cores.

[14] This selection of acceptable colleges nearly coincides with the edge-drawing procedure for 'small-world' networks presented by Watts and Strogatz (1998), letting $p = 1 - \ell/k$.

[15] In the Watts–Strogatz network model, k similarly controls the network density as well as the locality neighborhood radius.

2.4 Regular sequences of markets

Denote a *sequence of random markets* $\left(\tilde{\Gamma}^{(1)}, \tilde{\Gamma}^{(2)}, \dots\right)$, where each element $\tilde{\Gamma}^{(n)} = \left(C^{(n)}, S^{(n)}, \mathcal{P}_C^{(n)}, q^{(n)}, \mathcal{P}_S^{(n)}\right)$ is a random market in which $\left|C^{(n)}\right| = n$ is the number of colleges. A sequence of random markets is $(k, \overline{q})$-*regular* if there exist positive integers k and $\overline{q}$ such that:

- For all n, and all preference-orderings $\tilde{\succ}$ supported in $\mathcal{P}_S^{(n)}$, exactly k colleges are acceptable under $\tilde{\succ}$.
- $q_c^{(n)} \leq \overline{q}$ for $c \in C^{(n)}$ for all n.
- $\left|S^{(n)}\right| \leq \overline{q}n$ for all n.
- For all n and $c \in C^{(n)}$, every $s \in S^{(n)}$ is acceptable to c.[16]

Call a random market *one-to-one* if $q_c^{(n)} = 1$ for all $c \in C^{(n)}$. Call such a market *balanced* if $\left|S^{(n)}\right| = \sum_{c \in C^{(n)}} q_c^{(n)}$ and (p, r)-*unbalanced* if $\left|S^{(n)}\right| = r + p \sum_{c \in C^{(n)}} q_c^{(n)}$.

3 Results

I present a lower bound on core size and describe incentives to manipulate stable matching mechanisms in Watts–Strogatz random markets. Theorem 3.1 presents a special case with balanced, one-to-one markets and full locality, Theorem 3.3 relaxes to unbalanced, many-to-one markets with partial locality, and Corollary 3.5 gives a manipulability result. These results apply to a broader class of models with preference locality; in Appendix C, I discuss a more general preference locality condition that yields similar results.

[16] This definition can be compared to Definition 2 of Kojima and Pathak (2009), with the first condition modified (though not weakened) to accept the change of students' preferences from $k^{(n)}$ independent draws from $\mathcal{D}^{(n)}$ (there) to a draw from a distribution over preference-orderings $\mathcal{P}_S^{(n)}$ (here). However, all results presented in this work concern markets which are (p, r)-unbalanced (of which balanced markets are a special case) and Watts–Strogatz, which are stronger assumptions than the third and fourth conditions, respectively. While these two conditions are thus redundant in the context of the present work, I include them here for expositional ease, and in particular to clarify the compatibility between the two definitions.

3.1 Core size in large, one-to-one markets

Consider a sequence of random markets $\big(\tilde{\Gamma}^{(n)}\big)_{n\in\mathbb{N}}$. For a random market $\tilde{\Gamma}^{(n)} = \big(C^{(n)}, S^{(n)}, \mathcal{P}_C^{(n)}, q^{(n)}, \mathcal{P}_S^{(n)}\big)$, let $\alpha\big(\tilde{\Gamma}^{(n)}\big)$ denote the number of colleges in $C^{(n)}$ with multiple stable matches and let $\beta\big(\tilde{\Gamma}^{(n)}\big)$ denote the number of students in $S^{(n)}$ with multiple stable matches.

Theorem 3.1 (One-to-one, balanced Watts–Strogatz random markets exhibit non-vanishing cores)**.** *Fix $k \geq 2$. There exists $\Delta(k) > 0$ such that, given a $(k,1)$-regular sequence $\big(\tilde{\Gamma}^{(n)}\big)_{n\in\mathbb{N}}$ of balanced, one-to-one, Watts–Strogatz random markets with $\ell = k$:*

$$\liminf_{n\to\infty} \mathbb{E}\alpha\big(\tilde{\Gamma}^{(n)}\big)/n > \Delta(k). \tag{3.1}$$

$$\liminf_{n\to\infty} \mathbb{E}\beta\big(\tilde{\Gamma}^{(n)}\big)/n > \Delta(k). \tag{3.2}$$

Proof. Consider two students $s_1, s_2 \in S^{(n)}$ and two colleges $c_1, c_2 \in C^{(n)}$.[17] Let the event $E^{(n)}(s_1, s_2, c_1, c_2)$ denote the case where:

1. College c_1 prefers s_1 to s_2. Formally, $s_1 \succ_{c_1} s_2 \succ_{c_1} \varnothing$.

2. College c_2 prefers s_2 to s_1. Formally, $s_2 \succ_{c_2} s_1 \succ_{c_2} \varnothing$.

3. No students other than s_1 and s_2 find c_1 or c_2 acceptable. Formally, for all $s \in S^{(n)} \backslash \{s_1, s_2\}$, $\varnothing \succ_s c_1$ and $\varnothing \succ_s c_2$.

4. The colleges that s_1 finds most desirable are c_2 and c_1, in that order. Formally, for all $c \in C^{(n)} \backslash \{c_1, c_2\}$, $c_2 \succ_{s_1} c_1 \succ_{s_1} c$.

5. The colleges that s_2 finds most desirable are c_1 and c_2, in that order. Formally, for all $c \in C^{(n)} \backslash \{c_1, c_2\}$, $c_1 \succ_{s_2} c_2 \succ_{s_2} c$.

Note that, in the event $E^{(n)}(s_1, s_2, c_1, c_2)$, the four agents have two stable allocations: $\{(s_1, c_2), (s_2, c_1)\}$ (the student-optimal allocation), and $\{(s_1, c_1), (s_2, c_2)\}$ (the college-optimal allocation).

The first statement, $\liminf_{n\to\infty} \mathbb{E}\alpha\big(\tilde{\Gamma}^{(n)}\big)/n > \Delta(k)$, follows from Lemma 3.2 below, which places a positive lower bound[18] on the probability that a college

[17] The proof roughly follows that of Theorem 3 of Biró *et al.* (2022), who use a similar construction to bound below the fraction of agents with multiple stable matches, in their matching-with-contracts setting.

[18] Specifically, $\Delta(k) = \frac{\exp[-k-1]}{4k^2}$.

c_1 is involved in some such $E^{(n)}(s_1, s_2, c_1, c_2)$, and which holds for sufficiently large n.

The second statement, $\liminf_{n\to\infty} \mathbb{E}\beta(\tilde{\Gamma}^{(n)})/n > \Delta(k)$, then follows by a counting argument: each pair of colleges c_1, c_2 where an event $E^{(n)}$ occurs corresponds to exactly one pair of students s_1, s_2 with multiple stable matches. $\square$

Lemma 3.2. *Fix $k \geq 2$ and $\epsilon > 0$. There exists sufficiently large n such that for each college $c_1 \in C^{(n)}$ the event*

$$E_{c_1}^{(n)} := \bigcup_{(s_1,s_2,c_2)\in S^{(n)}\times S^{(n)}\times C^{(n)}} E^{(n)}(s_1, s_2, c_1, c_2) \tag{3.3}$$

has probability bounded below by $\Delta(k) := \frac{\exp[-k-1]}{4k^2} - \epsilon$.

Proof. Fix an arbitrary $c_1 \in C^{(n)}$. For different selections of (s_1, s_2, c_2), the events $E^{(n)}(s_1, s_2, c_1, c_2)$ are disjoint. There are $n\cdot(n-1)$ possible selections such that $s_1 \neq s_2$ and $y_{c_2} - y_{c_1} = 1$. The first and second requirements of the event definition are satisfied with probability $1/2$ each, the third with probability $\left(1 - \frac{k+1}{n}\right)^{n-2}$, and the fourth and fifth with probability $1/nk$ each. These probabilities are independent, as each concerns the preferences of distinct agents. Thus, the probability of the event $E_{c_1}^{(n)}$ is at least

$$n \cdot (n-1) \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \left(1 - \frac{k+1}{n}\right)^{(n-2)} \cdot \frac{1}{nk} \cdot \frac{1}{nk}. \tag{3.4}$$

The limit inferior of this expression as $n \to \infty$ is $\Delta(k) := \frac{\exp[-k-1]}{4k^2}$. $\square$

Remark 1. *The factors of $\frac{1}{nk}$ in equation 3.4 would each instead be $\frac{1}{n^2}$ in a uniform random market, where this technique would instead yield a vanishing $O\left(\frac{\exp[-k-1]}{4n^2}\right)$ lower bound.*

3.2 Core size in large, unbalanced markets

The results of Theorem 3.1 hold when generalizing one-to-one markets to markets with bounded college capacities and introducing a constant-fraction cross-side imbalance in agent populations. Furthermore, the results hold with high probability.

Theorem 3.3 (General Watts–Strogatz random markets exhibit non-vanishing cores)**.** *Fix $k \geq 2$ and $1 < \ell \leq k$. There exists $\Delta(k, \ell, p, \overline{q}) > 0$ such* *that, given a $(k,\overline{q})$-regular sequence $\left(\tilde{\Gamma}^{(n)}\right)_{n\in\mathbb{N}}$ of (p,r)-unbalanced, Watts–Strogatz random markets with locality parameter ℓ:*

$$\liminf_{n\to\infty} \mathbb{E}\alpha\left(\tilde{\Gamma}^{(n)}\right)/n > \Delta(k,\ell,p,\overline{q}) \tag{3.5}$$

$$\liminf_{n\to\infty} \mathbb{E}\beta\left(\tilde{\Gamma}^{(n)}\right)/n > \Delta(k,\ell,p,\overline{q}). \tag{3.6}$$

Theorem 3.4 (Theorem 3.3 applies with high probability)**.** *Fix $1 < \ell \leq k \geq 2$ and $\epsilon > 0$. There exists $\Delta_\epsilon(k,\ell,p,\overline{q}) > 0$ such that, given a $(k,\overline{q})$-regular sequence $\left(\tilde{\Gamma}^{(n)}\right)_{n\in\mathbb{N}}$ of (p,r)-unbalanced, Watts–Strogatz random markets with locality parameter ℓ, for sufficiently large n:*

$$\Pr[\alpha\left(\tilde{\Gamma}^{(n)}\right)/n > \Delta_\epsilon(k,\ell,p,\overline{q})] > 1-\epsilon. \tag{3.7}$$

$$\Pr[\beta\left(\tilde{\Gamma}^{(n)}\right)/n > \Delta_\epsilon(k,\ell,p,\overline{q})] > 1-\epsilon. \tag{3.8}$$

Appendix A presents both proofs by a similar construction to Theorem 3.1, modified so that c_1's quota is filled and c_2's rejection of s_1 forces c_1 to reject some other student s_2 in turn.

3.3 Incentives to manipulate in large, unbalanced markets

Fix some core-selecting mechanism and consider a sequence of random markets $\left(\tilde{\Gamma}^{(n)}\right)_{n\in\mathbb{N}}$. For a random market $\tilde{\Gamma}^{(n)}$, let $\gamma\left(\tilde{\Gamma}^{(n)}\right)$ denote the number of agents who can improve their assigned outcomes by misreporting their true preferences (when all others report their true preferences).

Corollary 3.5 (A non-vanishing fraction of agents can manipulate DA in a Watts–Strogatz random market)**.** *Fix $1 < \ell \leq k \geq 2$. Given a $(k,\overline{q})$-regular sequence $\left(\tilde{\Gamma}^{(n)}\right)_{n\in\mathbb{N}}$ of (p,r)-unbalanced, Watts–Strogatz random markets with locality parameter ℓ, there exists $\Delta(k,\ell,p,\overline{q}) > 0$ such that*

$$\liminf_{n\to\infty} \mathbb{E}\gamma\left(\tilde{\Gamma}^{(n)}\right)/n > \Delta(k,\ell,p,\overline{q}). \tag{3.9}$$

Proof. Any agent with multiple stable matches can select a match among them by misreporting their true preferences (Gale and Sotomayor (1985)). Of the agents with multiple stable matches in Theorem 3.3, no core-selecting mechanism assigns both sides of a college–student group their favored stable matches. So

$$\gamma\left(\tilde{\Gamma}^{(n)}\right) \geq \min\left\{\alpha\left(\tilde{\Gamma}^{(n)}\right), \beta\left(\tilde{\Gamma}^{(n)}\right)\right\}, \tag{3.10}$$

and the expected fraction is at least the $\Delta(k,\ell,p,\overline{q})$ from Lemma A.2. □

4 Computational experiments

The theoretical results of the previous section prove that core size (as a fraction of the market) will be non-vanishing in the large market limit. However, the theoretical bounds presented are not substantial in any practical sense—in Theorem 3.2, $\Delta(k) \approx 0.05\%$ for $k = 3$. In this section, I present simulation results that complement those theoretical results by demonstrating the realized size of large cores in simulated Watts–Strogatz random markets with full locality. In general, I find that core size exceeds 4% in practice under a wide variety of specifications of locality, cross-side imbalance, and size.

My approach to these computation experiments follows Ashlagi *et al.* (2017), replicating their results on core convergence under cross-side imbalance in $|C| = k = 40$ market specifications. I then present analogous results under models of preferences with locality for $|C| = 40$ and various k, as well as larger sizes of $|C|$. For each market specification, I simulate a number of realizations by drawing random preferences independently for each agent, and computing the stable matching optimal for each side of the market.

Additional figures in Appendix B demonstrate that core size and welfare gaps are roughly unchanged from $|C| = 400$ through $|C| = 4000$ as k remains constant, and that they remain apparent through $q = 10$ and beyond.

4.1 Strategic incentives in unbalanced markets

The first experiment illustrates that preference locality can support scope for strategic incentives and attenuate the sharp welfare effect of cross-side imbalance in a small market. I specify a Watts–Strogatz random market with 40 colleges, between 20 and 80 students, and $k = \ell \in \{3, 5, 10, 20, 40\}$ across 30,000 realizations.

Figure 1 reports the fraction of matched agents who have multiple stable matches; this fraction is small in unbalanced markets under homogeneous ($k = 40$) preferences, but substantial for even substantially unbalanced markets where $k = 3$, 5, or 10.

Figure 2 reports averages across realizations of the matched students' rank of matches under the student-optimal and student-pessimal stable matchings. The results for $k = 40$ replicate prior work by Ashlagi *et al.* (2017), who describe the homogeneous case:

> [I]n any unbalanced market, the men's average rank of wives is almost the same under the [men-optimal stable matching] and [women-optimal stable matching]. When there are fewer men than women (*i. e.*, fewer than 40 men), the men's average rank

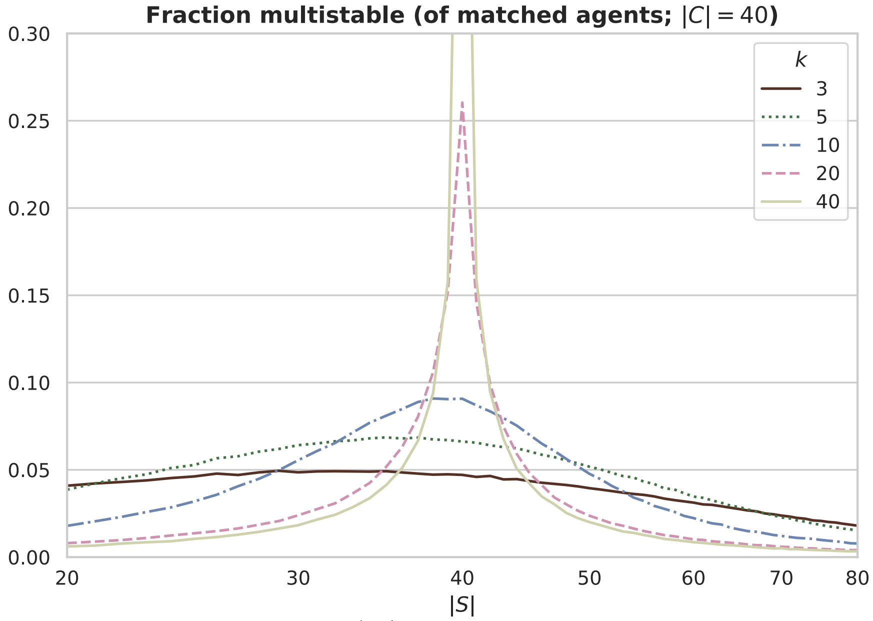

Figure 1: Average core size for $|C| = 40$, $k = \ell$, and $q = 1$. Compare $k = 40$ to Figure 1 in Ashlagi *et al.* (2017).

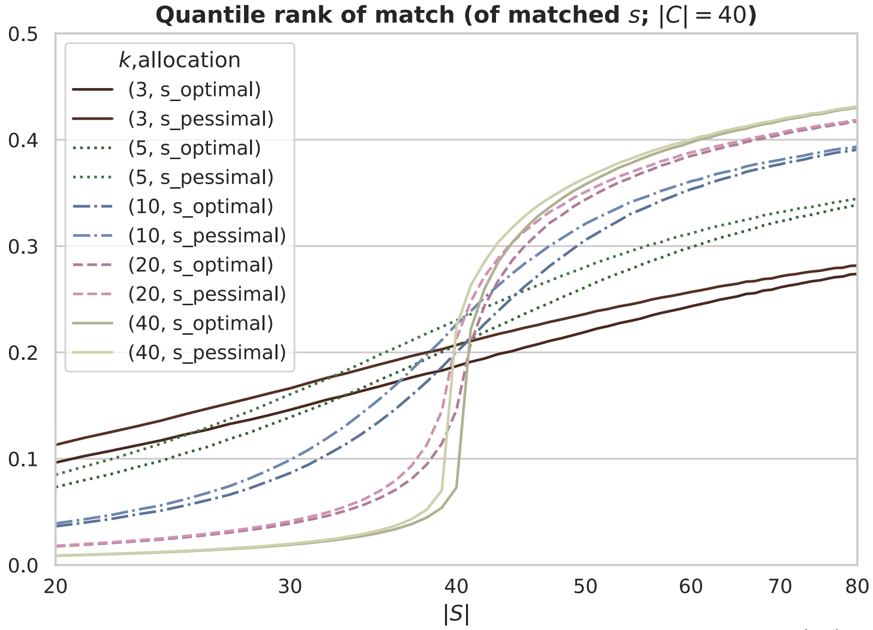

Figure 2: Matched students' average quantile rank of matches for $|C| = 40$, $k = \ell$, and $q = 1$. Compare $k = 40$ to Figure 2 in Ashlagi *et al.* (2017).

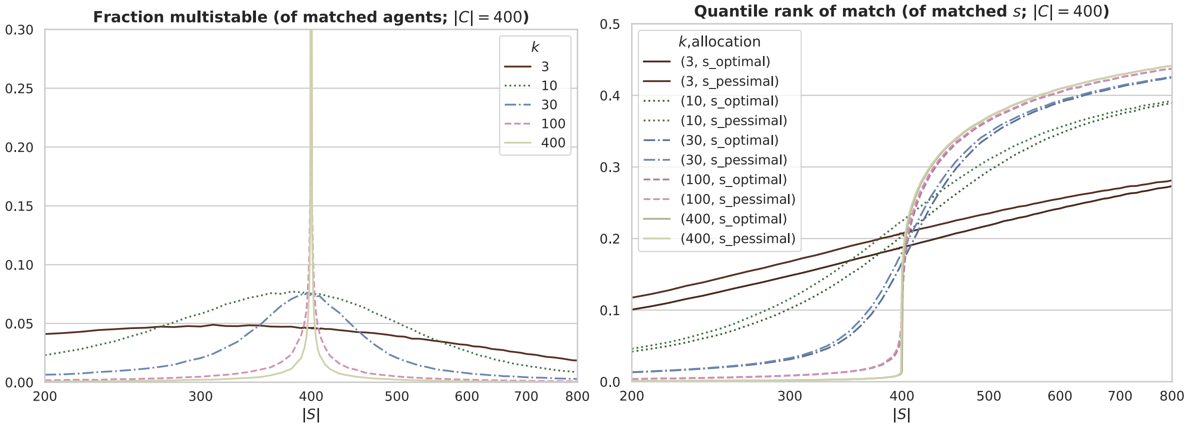

Figure 3: Average core size and matched students' average quantile rank of matches for $|C| = 400$, $k = \ell$, and $q = 1$.

> of wives under any stable matching is almost the same as under [random serial dictatorship], with most men receiving one of their top choices. When there are more men than women in the market, the men's average rank of wives is not much better than 20.5 [of 40], which would be the result of a random assignment.

But under more-localized (*i. e.*, smaller-k) preferences, three differences are apparent:

- The students' welfare gap between optimal and pessimal matches is substantially larger in unbalanced markets.
- The students' welfare gap does not increase sharply as markets approach balance.
- Students' absolute welfare (under any stable matching) depends less sharply on the cross-side imbalance.

Colleges' intra-core welfare gaps and absolute welfare with respect to cross-side imbalance respond similarly to preference locality; see Figures 8 and 9 in Appendix B.

4.2 Core size and allocation outcomes in large markets

I also present results of simulations of large unbalanced matching markets under various specifications. Figure 3 reports the fraction of matched agents who have multiple stable matches and matched students' average rank of matches (across 3,000 realizations) in a market with 400 colleges, between 200 and 800 students, and preferences given by the Watts–Strogatz model for $k = \ell \in \{3, 10, 30, 100, 400\}$.

In general, as in the smaller specification, the size of the core and the strategic incentives do decrease as the market becomes unbalanced, but much more slowly under more-localized (*i. e.*, smaller-k) preferences than under the homogeneous specification. As an example: in a market with 440 students, 400 colleges, and $k = 30$, more than 5% of matched students have more than one stable match, and among students with more than one stable match, the average rank difference between their optimal and pessimal match is greater than 7 (*i. e.*, nearly a quarter of their preference-list length). In the homogeneous specification, the incentives are only an eighth as large. In a more unbalanced market, the effect is even more stark—with 500 students, 400 colleges, and $k = 10$, more than 5% of matched students have more than one stable match, but with $k = 400$, fewer than 0.24% do.

I present further simulation results for markets with $q|C| \in \{400, 1000, 4000\}$ college-seats, and for $q \in \{1, 4, 10\}$, in Figures 10–12 in Appendix B. The gap in student welfare between the student-optimal and student-pessimal allocations remains apparent, and the fraction of agents with more than one stable match fails to vanish, as the market size grows.

4.3 Localized versus incomplete uniform preferences

While locality in these experiments is accompanied by shortened list lengths, the core-size results do not come from short lists alone. By setting an independent probability that a student s finds a college c acceptable to $k/|C|$, I can specify incomplete uniform preferences under which students are on average as selective as in a Watts–Strogatz model of equal k.

Figures 4 and 5 report core size and matched students' average rank of matches under such a preference model (across 10,000 realizations), and under a Watts–Strogatz model for $|C| = 40$. Under incomplete uniform random preferences, shortening list length reduces core size[19] and yields a small intra-core welfare gap. Given a Watts–Strogatz preference model, however, shorter lists create smaller cores in balanced markets but larger cores in markets with sufficient cross-side imbalance, and a substantially larger intra-core welfare gap is supported in low-k conditions.

[19] Compare Figure 2 in Roth and Peranson (1999), which reports core size decreasing with shorter preference lists in simulation experiments.

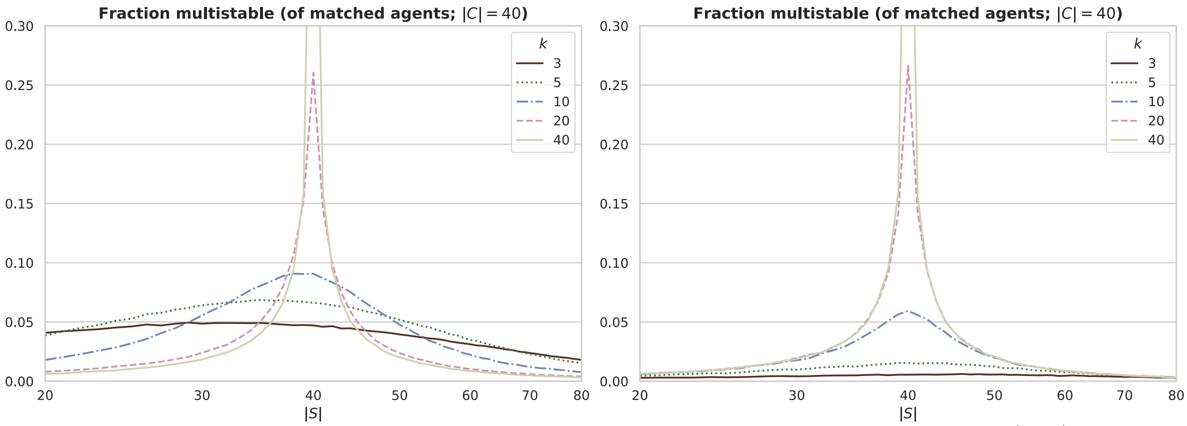

Figure 4: Average core size under $k = \ell$ Watts–Strogatz preferences (left) and similarly-selective uniform random preferences (right) with $q = 1$. Plot at left reproduces Figure 1.

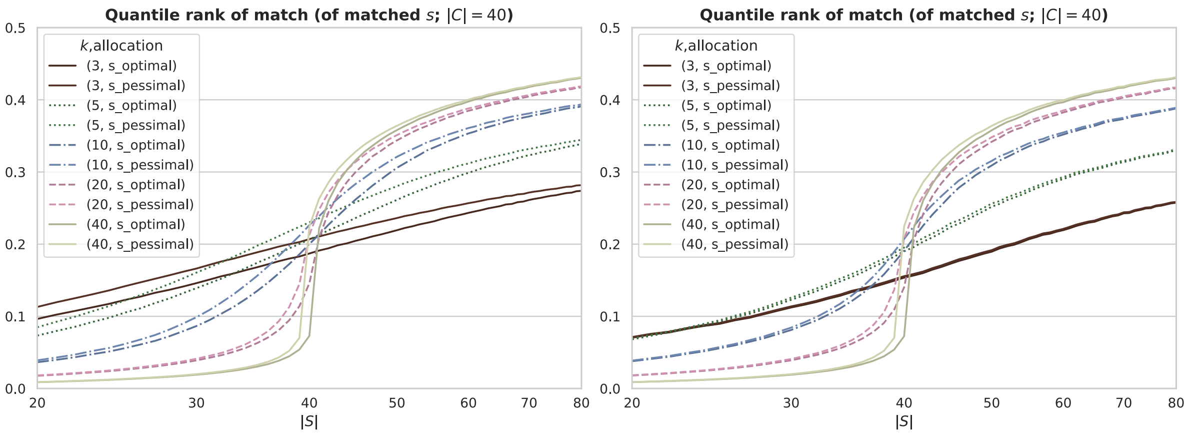

Figure 5: Matched students' average quantile rank of matches under $k = \ell$ Watts–Strogatz preferences (left) and similarly-selective uniform random preferences (right) with $q = 1$. Plot at left reproduces Figure 2.

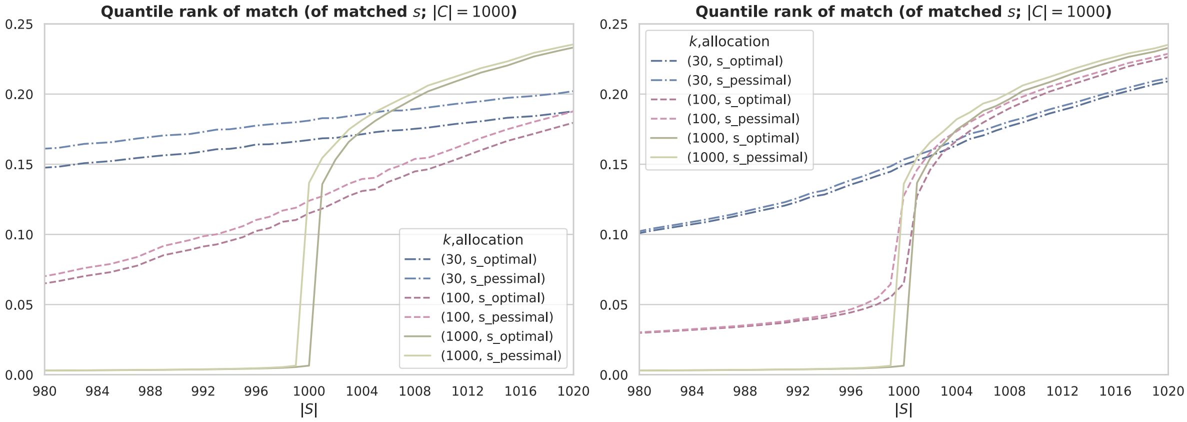

Figure 6: Matched students' average quantile rank of matches under $k = \ell$ Watts–Strogatz preferences (left) and similarly-selective incomplete uniform preferences (right) with $q = 1$, in detail around $|S| \approx |C|$. Note that Kanoria *et al.* (2021) suggest that $k \sim \log(1000)^2 \approx 48$ is the density threshold above which a stark effect of competition should be exhibited by an incomplete uniform random market.

One effect which is visually apparent in both preference models is that matched students' absolute welfare depends less sharply on the cross-side imbalance under lower-k conditions. This aligns with the simulation results of Ashlagi *et al.* (2017) and theoretical results of Kanoria *et al.* (2021).

Kanoria *et al.* (2021) further suggest $k \sim \log^2(n)$ as the threshold above which an incomplete uniform random market will exhibit a stark effect of competition. However, further experiments show that this result is dependent on the uniformity of agent preferences, and that even at densities above that threshold, locality can serve to blunt the effect of cross-side imbalance. As an example, Figure 6 reports detail on matched students' average rank of match for $|C| = 1000$. In a Watts–Strogatz random market with $k = \ell = 100$ (across 1,000 realizations), the effect of competition is fairly smooth through the range of $|S|/|C| \in [98\%, 102\%]$; this is well above the Kanoria *et al.* (2021) threshold level of $k \sim 48$. In an incomplete uniform random market (across 150 realizations), a sharp effect of competition becomes apparent to visual inspection between the $k = 30$ and $k = 100$ conditions. While the Watts–Strogatz specification has a modestly greater fraction of agents unmatched (see Figure 7), the unmatched fraction in the $k = 100$ condition (less than 0.8%) is not significant enough to drive this effect.

Results of further simulations of incomplete uniform random markets for $|C| = 400$ and $|C| = 1000$ are presented in Figure 13 in Appendix B. Fig-

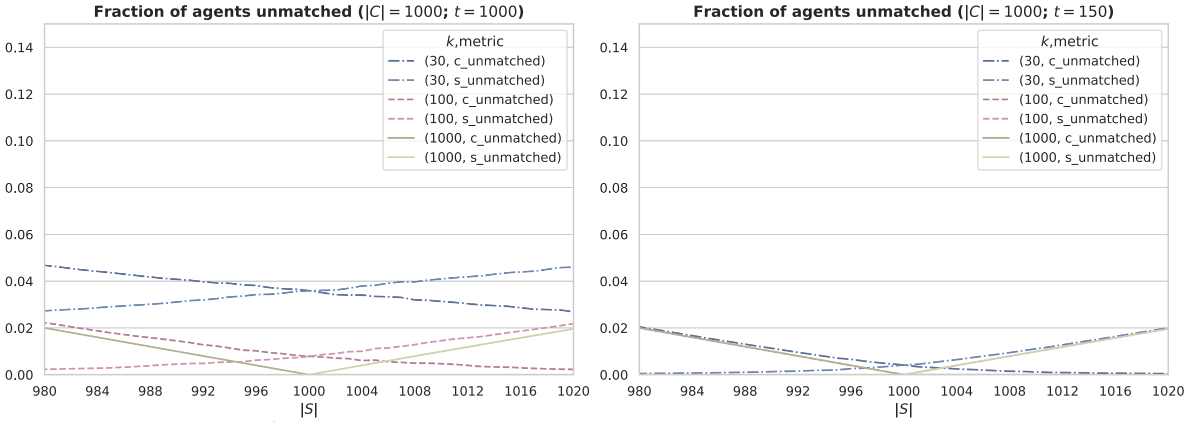

Figure 7: Average fraction of unmatched agents under $k = \ell$ Watts–Strogatz preferences (left) and similarly-selective incomplete uniform preferences (right) with $q = 1$, in detail around $|S| \approx |C|$.

ure 14 reports the fraction of unmatched agents for all experiments presented in this section.

5 Discussion

It is beyond the scope of the present work to present a comprehensive structural theory of matching markets. However, I discuss some empirical observations on matching markets and ways in which locality may interact with other structural properties of matching markets, presenting some questions for further investigation.

5.1 Small cores in large markets

In apparent contrast to the present work, some empirical studies of matching markets have found small cores in large markets. For example, Roth and Peranson (1999) examined the National Resident Matching Program (NRMP)'s market for medical residency positions from 1993 to 1996 and found that in a market with roughly 20,000 applicants and potential positions, only about 0.1% of residents were assigned different matches by primarily resident-proposing and primarily program-proposing mechanisms.[20]

[20]Neither the baseline NRMP mechanism nor the mechanism redesigned by Roth and Peranson (1999) were simple deferred-acceptance mechanisms, so this comparison is not completely within the theoretical framework of the present work. Nevertheless, the mechanisms in question were both based heavily on deferred-acceptance mechanisms, and did

This is a significant divergence from the roughly 4% that might be predicted by some models presented in the present work.

These small cores are not due to match variations[21] in the NRMP match—in the same work, the authors presented five years of data on a matching market for thoracic surgery residents with roughly 175 applicants (130 positions) and no match variations, in which fewer than 0.5% of residents had more than one stable match; some models of the present work might suggest rates ten times that, given tightly localized preferences. Similarly, Kojima *et al.* (2013) examined a matching market for clinical psychologists (with roughly 3,000 applicants[22] and 2,700 positions) from 1999 to 2007 and found that roughly 0.2% of residents had more than one stable match. Pathak and Sönmez (2008) examined two years of matching markets for primary- and secondary-school admissions (with roughly $3,000$ applicants at each level per year) in the Boston Public Schools and found just five instances of applicants with multiple stable matches.

5.2 Preferences and core size

It is possible that agents in the settings discussed above do not have preferences that exhibit sufficient locality to produce non-negligible cores. However, it is also possible that agents' preferences do exhibit locality, but other structures in their preferences induce the observed core convergence regardless. The most intuitively likely such structure is global alignment of preferences on one or both sides of the market; competition effects from locally-relevant cross-side imbalance may also contribute.

5.2.1 Preference alignment and core convergence

It is well-known that when preferences of agents on one side of the market align with a single complete ranking, only one stable matching exists. Moreover, Holzman and Samet (2014) show that the size of the core and the intra-core welfare gap can be bounded in terms of the disagreement in preference rankings between agents on the same side of the market. Even

differ in whether residents or programs proposed in the primary stage. Comparable results obtained when the same authors investigated the fraction of hospitals with incentives to manipulate the redesigned, applicant-proposing mechanism.

[21] Roth and Peranson (1999) identify four specific "match variations"(*i. e.*, deviations from the doctor-proposing deferred acceptance mechanism) in the NRMP match, which were introduced to accommodate institutional goals.

[22] The cited authors removed pairs of applicants registered as couples (roughly 19 per year) for the cited experiment.

if agents' preferences do not completely align, global alignment shrinks the core.

Remark 2. *While the Watts–Strogatz preference model of the present work can be said to have preferences "correlated" between agents, it is relevant that there is no global alignment. Rather, each pair of agents in this model will on average have only as much agreement in their preference lists as do two agents under uniform random preferences. This can be reconciled with an intuitive sense that Watts–Strogatz preferences are more "correlated" than uniform preferences by observing that agents' preferences align with other agents' preferences,* conditional on aligning at least somewhat*, but are not more aligned* on average*. This is unlike forms of global correlation studied in prior work, which have been shown to cause core convergence (Holzman and Samet (2014); Ashlagi* et al. *(2017)) and reduced scope for manipulation (Coles and Shorrer (2014); Coles* et al. *(2014)).*

It is intuitive that widely-agreed-upon variation in institutional quality serves to align students' preferences in residency matches and school-choice matches alike. Indeed, Kojima *et al.* (2013) found significant concentration of program popularities in the clinical psychology match, with a small number of programs receiving as many as eight times the number of first-place rankings that would be predicted by uniform draws.[23]

5.2.2 Other preference structures and core convergence

Furthermore, it is possible that the effect of competition (Ashlagi *et al.* (2017); Kanoria *et al.* (2021)) applies locally to cause the core to converge, even if the overall market is well-balanced. For example, Roth and Peranson (1999) find that there are somewhat more doctors than total hospital quota in the 1993–1996 NRMP market, yet the very naming of the "rural hospitals theorem" (Roth (1986))[24] indicates that there is an identifiable class of hospitals where total quota meaningfully exceeds the number of interested

[23] The authors further note that "these are preferences stated after interviews have been conducted, so [they do] not preclude the possibility that there are popular programs that receive many applications but only interview a small subset of applicants", and further that "an applicant typically ranks a program only after she interviews at the program, and each applicant receives and can travel to only a limited number of interviews." The true degree of concentration of popularity may be much greater.

[24] The *rural hospitals theorem* (Roth (1986)) states that agents not matched in one stable matching are not matched in any stable matching. Its presentation was motivated by the observation that rural hospitals would frequently fail to fill their available positions in the NRMP.

applicants. If there are sufficiently more rural hospitals than doctors interested in a rural residency, and sufficiently more urban-focused doctors than urban hospitals, all agents may face a local effect of competition that causes core convergence everywhere.

Finally, incompletely reported preferences may reduce the size of the apparent core by falsely excluding matches that would be stable with respect to true preferences, but do not appear among reported preferences. Agents may shorten preference lists when they face administrative constraints, costs to search (Shorrer (2019)), or strategically truncate to manipulate matching outcomes (Mongell and Roth (1991); Coles and Shorrer (2014)).

5.2.3 Locality and market-size effects on the core

Locality—in the sense of latent features that cause agents' preferences to vary heterogeneously over local clusters of like partners—is likely present in some sense in most matching markets. In the clinical psychology match, for example, Kojima *et al.* (2013) find geographic locality in applicant preferences, with half of single applicants ranking programs in at most two of eleven geographical regions. Beyond national geography, institutional features such as operational style or specialty focus may induce locality among applicant preferences. In school-choice, rideshare, or social matching settings, geographic locality within a city or region may be important as well.

While observations of small cores are common in settings of school choice and doctor residency, it is not clear whether market designers should conclude from them that market size and global competition effects form a complete or universal explanation for core convergence. Especially given that observed core convergence is near-total in these settings, it may be that preference alignment, local competition effects, and constraints that lead to shorter preference lists may play key roles in explaining core size, and it is not clear what to assume about the cores of large markets where these factors are weaker or absent. Determining a comprehensive explanation for small cores will thus require analysis of more than just market size and preference-list length; characterization of preference structures that expand or shrink the core will produce a more complete picture of these markets.

5.3 Conclusion

This work presents an opportunity at the intersection of two increasingly relevant topics—design of mechanisms for large matching markets and structural properties of large networks—to better understand the real-world settings addressed by the market-design literature. By considering the matching-market analogue of a canonically simple model of network structure, I find effects on welfare and incentives not supported by homogeneous market models in prior work.

In exploring the consequences of these results, I join a tradition of inquiry that aims to understand why we empirically observe such small cores in real-world matching markets, a question first posed by Roth and Peranson (1999). Those and some subsequent authors have proposed that such observations may be straightforwardly explained as a consequence of the large size of the markets in question, and of agents' relatively short preference lists. The results of the present work suggest that, if core convergence is indeed driven by market size, then that fact relies on student and college preferences being sufficiently homogeneous. If they are not, then preference alignment, competition effects, and limitations on preference-list length may play important roles in core convergence instead. Whether these small cores are in fact caused by short and well-mixed preferences, or rely on other properties of these markets, remains unresolved.

On this and other questions, I anticipate scope for future work that uses tools from the network-structure literature to characterize the structures of preferences found in large markets of interest and the effects of such structures on matching outcomes and incentives.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the author used Anthropic Claude and OpenAI ChatGPT in order to revise and refine expository elements. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

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Online Appendices

A Proofs excluded from main text

The results of Theorem 3.1 hold when generalizing one-to-one markets to markets with bounded college capacities and introducing a constant-fraction cross-side imbalance in agent populations. This proof proceeds using an appropriately modified construction and a revised asymtotic bound.

A.1 Core size in large, unbalanced markets

Consider a sequence of random markets $\big(\tilde{\Gamma}^{(n)}\big)_{n\in\mathbb{N}}$. For a random market $\tilde{\Gamma}^{(n)}$, let $\alpha\big(\tilde{\Gamma}^{(n)}\big)$ denote the number of colleges with multiple stable matches and let $\beta\big(\tilde{\Gamma}^{(n)}\big)$ denote the number of students with multiple stable matches.

Theorem A.1 (Main text: Theorem 3.3)**.** *Fix $k \geq 2$ and $1 < \ell \leq k$. There exists $\Delta(k,\ell,p,\overline{q}) > 0$ such that, given a $(k,\overline{q})$-regular sequence $\big(\tilde{\Gamma}^{(n)}\big)_{n\in\mathbb{N}}$ of (p,r)-unbalanced, Watts–Strogatz random markets with locality parameter ℓ:*

$$\liminf_{n\to\infty} \mathbb{E}\alpha\big(\tilde{\Gamma}^{(n)}\big)/n > \Delta(k,\ell,p,\overline{q}). \tag{A.1}$$

$$\liminf_{n\to\infty} \mathbb{E}\beta\big(\tilde{\Gamma}^{(n)}\big)/n > \Delta(k,\ell,p,\overline{q}). \tag{A.2}$$

Proof. Consider two colleges $c_1, c_2 \in C^{(n)}$ and students $s_1, \ldots, s_{q+1} \in S^{(n)}$, where $q := q_{c_1}$. Let the event $E^{(n)}(s_1, \ldots, s_{q+1}, c_1, c_2)$ denote the case where:

1. College c_1 ranks s_2 last of $s_1, \ldots, s_{q+1}$. Formally, for all $s \in \{s_1, s_3, \ldots, s_{q+1}\}$, $s \succ_{c_1} s_2 \succ_{c_1} \varnothing$.
2. College c_2 ranks s_2 higher than s_1. Formally, $s_2 \succ_{c_2} s_1 \succ_{c_2} \varnothing$.
3. No students other than $s_1, \ldots, s_{q+1}$ find c_1 or c_2 acceptable. Formally, for all $s \in S^{(n)} \backslash \{s_1, \ldots, s_{q+1}\}$, $\varnothing \succ_s c_1$ and $\varnothing \succ_s c_2$.
4. The colleges that s_1 finds most desirable are c_2 and c_1, in that order. Formally, for all $c \in C^{(n)} \backslash \{c_1, c_2\}$, $c_2 \succ_{s_1} c_1 \succ_{s_1} c$.
5. The colleges that s_2 finds most desirable are c_1 and c_2, in that order. Formally, for all $c \in C^{(n)} \backslash \{c_1, c_2\}$, $c_1 \succ_{s_2} c_2 \succ_{s_2} c$.
6. The college that $s_3, \ldots, s_{q+1}$ find most desirable is c_1. Formally, for all $s \in \{s_3, \ldots, s_{q+1}\}$ and $c \in C^{(n)} \backslash \{c_1\}$, $c_1 \succ_s c$.

Note that, in the event $E^{(n)}(s_1,\ldots,s_{q+1},c_1,c_2)$, the agents have two stable allocations:

- $\{(s_1,c_2),(s_2,c_1),(s_3,c_1),\ldots,(s_{q+1},c_1)\}$ (the student-optimal allocation)
- $\{(s_1,c_1),(s_2,c_2),(s_3,c_1),\ldots,(s_{q+1},c_1)\}$ (the college-optimal allocation).

The first statement, $\liminf_{n\to\infty}\mathbb{E}\alpha\big(\tilde\Gamma^{(n)}\big)/n > \Delta(k,\ell,p,\overline{q})$, follows from Lemma A.2 below, which places a positive lower bound[25] on the probability that a college c_1 is involved in some such $E^{(n)}(s_1,\ldots,s_{q+1},c_1,c_2)$, and which holds for sufficiently large n.

The second statement, $\liminf_{n\to\infty}\mathbb{E}\beta\big(\tilde\Gamma^{(n)}\big)/n > \Delta(k,\ell,p,\overline{q})$, then follows by a counting argument: each pair of colleges c_1,c_2 where an event $E^{(n)}$ occurs corresponds to exactly one pair of students s_1,s_2 with multiple stable matches. $\square$

Lemma A.2. *Fix $1<\ell\le k\ge 2$ and $\epsilon>0$. There exists sufficiently large n such that for each college $c_1\in C^{(n)}$ the event*

$$E^{(n)}_{c_1} := \bigcup_{(s_1,\ldots,s_{q+1},c_2)\in\left(S^{(n)}\right)^{\times(q+1)}\times C^{(n)}} E^{(n)}(s_1,\ldots,s_{q+1},c_1,c_2) \tag{A.3}$$

has probability bounded below by $\Delta(k,\ell,p,\overline{q}) := \frac{\ell^4\min(1,p)^{(\overline{q}+1)}\exp[-p\overline{q}(3k-2\ell+1)]}{4k^6\overline{q}!}$.

Proof. Consider an arbitrary $c_1\in C^{(n)}$. For different selections of $(s_1,\{s_2,\ldots,s_{q+1}\},c_2)$, the events $E^{(n)}(s_1,\ldots,s_{q+1},c_1,c_2)$ are disjoint. There are $\left|S^{(n)}\right|\cdot\left({}_{(|S^{(n)}|-1)}C_q\right)$ possible selections of distinct $(s_1,\{s_2,\ldots,s_{q+1}\})$ and $y_{c_2}-y_{c_1}=1$. Let s_2 be the c_1-dispreferred student among $\{s_2,\ldots,s_{q+1}\}$ without loss of generality. The probability of each such $E^{(n)}(s_1,\ldots,s_{q+1},c_1,c_2)$ is at least

$$\frac{1}{2}\cdot\frac{1}{2}\cdot\left(1-\frac{k+1}{n}-\frac{2(k-\ell)}{n-\ell}\right)^{(|S^{(n)}|-q-1)}\cdot\frac{\ell^2}{nk^3}\cdot\frac{\ell^2}{nk^3}\cdot\left(\frac{1}{n}\right)^{(q-1)}, \tag{A.4}$$

where each term in this expression corresponds to a requirement[26] from the definition of $E^{(n)}(s_1,\ldots,s_{q+1},c_1,c_2)$, so long as $\ell\ge 2$. These probabilities

[25] Specifically, $\Delta(k,\ell,p,\overline{q}) = \frac{\ell^4\min(1,p)^{(\overline{q}+1)}\exp[-p\overline{q}(3k-2\ell+1)]}{4k^6\overline{q}!}$, or more strongly $\Delta(k,\ell,p,\overline{q}) = \frac{p\ell^4\cdot\exp[-p\overline{q}(3k-2\ell+1)]}{4k^6}$ when $\liminf_{n\to\infty}\left|S^{(n)}\right|/n>1$, as demonstrated in Lemma A.3.

[26] To be precise, the first, second, third, fourth, fifth, and sixth requirements, respectively, where the first requirement requires only $s_1\succ_{c_1}s_2$, allowing s_2 to be the c_1-dispreferred student among $\{s_2,\ldots,s_{q+1}\}$ without loss of generality.

are independent, as each concerns the preferences of distinct agents. Thus, the probability of the event $E^{(n)}_{c_1}$ is at least

$$\Delta^{(n)}_{(k,\ell,p,\overline{q})} := \left|S^{(n)}\right| \cdot \left(\prod_{i=1}^{q} \frac{\left|S^{(n)}\right| - 1 - i}{i}\right) \cdot \frac{\left(1 - \frac{k+1}{n} - \frac{2(k-\ell)}{n-\ell}\right)^{\left(|S^{(n)}|-q-1\right)} \cdot \ell^2 \cdot \ell^2}{4 \cdot nk^3 \cdot nk^3 \cdot n^{(q-1)}}. \tag{A.5}$$

Let $\overline{Q} := \limsup_{n\to\infty} \left|S^{(n)}\right|/n$ and $\underline{Q} := \liminf_{n\to\infty} \left|S^{(n)}\right|/n$; then

$$\liminf_{n\to\infty} \Delta^{(n)}_{(k,\ell,p,\overline{q})} \geq \frac{\underline{Q}^{(q+1)} \cdot \exp\left[-\overline{Q}(3k - 2\ell + 1)\right] \cdot \ell^4}{4k^6 q!}. \tag{A.6}$$

So, noting $\overline{Q} \leq p\overline{q}$ and $\underline{Q} \geq p$, conclude

$$\Pr\left[E^{(n)}_{c_1}\right] \geq \Delta(k,\ell,p,\overline{q}) := \frac{\ell^4 \min(1,p)^{(\overline{q}+1)} \exp[-p\overline{q}(3k - 2\ell + 1)]}{4k^6 \overline{q}!} \tag{A.7}$$

for sufficiently large n. Lemma A.3 shows that a tighter bound may be obtained instead when $\underline{Q} > 1$. □

Lemma A.3. *Fix $1 < \ell \leq k \geq 2$ and $\epsilon > 0$. If $\underline{Q} > 1$, then there exists sufficiently large n such that for each college $c_1 \in C^{(n)}$ the event $E^{(n)}_{c_1}$ has probability bounded below by $\Delta(k,\ell,p,\overline{q}) := \frac{p\ell^4 \exp[-p\overline{q}(3k-2\ell+1)]}{4k^6}$.*

Proof. If $\underline{Q} > 1$, then $\left|S^{(n)}\right| > n + \overline{q} + 1$ for sufficiently large n. Then $\left({}_{(|S|-1)}C_q\right) > n^q$, so

$$\liminf_{n\to\infty} \Delta^{(n)}_{(k,\ell,p,\overline{q})} \geq \frac{\underline{Q} \cdot \exp\left[-\overline{Q}(3k - 2\ell + 1)\right] \cdot \ell^4}{4k^6} \tag{A.8}$$

and

$$\Pr\left[E^{(n)}_{c_1}\right] \geq \Delta(k,\ell,p,\overline{q}) := \frac{p\ell^4 \exp[-p\overline{q}(3k - 2\ell + 1)]}{4k^6} \tag{A.9}$$

for sufficiently large n. □

Theorem A.4 (Main text: Theorem 3.4)**.** *Fix $1 < \ell \leq k \geq 2$ and $\epsilon > 0$. There exists $\Delta_\epsilon(k,\ell,p,\overline{q}) > 0$ such that, given a $(k,\overline{q})$-regular sequence $\left(\tilde{\Gamma}^{(n)}\right)_{n\in\mathbb{N}}$ of (p,r)-unbalanced, Watts–Strogatz random markets with locality parameter ℓ, for sufficiently large n:*

$$\Pr\left[\alpha\left(\tilde{\Gamma}^{(n)}\right)/n > \Delta_\epsilon(k,\ell,p,\overline{q})\right] > 1 - \epsilon. \tag{A.10}$$

$$\Pr\left[\beta\left(\tilde{\Gamma}^{(n)}\right)/n > \Delta_\epsilon(k,\ell,p,\overline{q})\right] > 1 - \epsilon. \tag{A.11}$$

Proof. Define the event $E^{(n)}\left(s_1^i, \ldots, s_{q_{c_i}+1}^i, c_i, c_i^*\right)$ as in Theorem A.1. The first statement follows from Lemma A.5 below, which shows that, with probability at least $1-\epsilon$, at least $n \cdot \Delta_\epsilon(k, \ell, p, \overline{q})$ colleges are involved in some $E^{(n)}\left(s_1^i, \ldots, s_{q_{c_i}+1}^i, c_i, c_i^*\right)$, and which holds for sufficiently large n.

The second statement then follows by a counting argument, as in Theorem A.1. $\square$

Lemma A.5. *Fix $1 < \ell \leq k \geq 2$ and $\epsilon > 0$. There exists sufficiently large n such the event*

$$E_{c_i}^{(n)} := \bigcup_{\left(s_1^i, \ldots, s_{q_{c_i}+1}^i, c_i^*\right) \in \left(S^{(n)}\right)^{\times (q_{c_i}+1)} \times C^{(n)}} E^{(n)}\left(s_1^i, \ldots, s_{q_{c_i}+1}^i, c_i, c_i^*\right) \quad \text{(A.12)}$$

occurs for at least

$$n \cdot \Delta_\epsilon(k, \ell, p, \overline{q}) := n \cdot \left(\frac{\ell^4 \min(1,p)^{(\overline{q}+1)} \exp[-p\overline{q}(3k - 2\ell + 1)]}{4k^6(k+1)\overline{q}!} \cdot (1-\epsilon)^4 \right) \quad \text{(A.13)}$$

of the $c_i \in C^{(n)}$ with probability bounded below by $1-\epsilon$.

Proof. Let $m := \lfloor n/(k+1) \rfloor$ and consider an arbitrary $\{c_1, c_2, \ldots, c_m\}$ with $\left|y_{c_i} - y_{c_j}\right| > k$ for $i \neq j$. For different selections of $\left(s_1^i, \left\{s_2^i, \ldots, s_{q_{c_i}+1}\right\}, c_i^*\right)$ the events $E^{(n)}\left(s_1^i, s_2^i, c_i, c_i^*\right)$ are disjoint. For each c_i, there are $\left|S^{(n)}\right| \cdot \left({}_{(|S^{(n)}|-1)}C_{q_{c_i}}\right)$ possible selections of distinct $\left(s_1^i, \left\{s_2^i, \ldots, s_{q_{c_i}+1}^i\right\}\right)$ and $y_{c_i^*} - y_{c_i} = 1$. Let $\overline{Q} := \limsup_{n\to\infty} \left|S^{(n)}\right|/n$ and $\underline{Q} := \liminf_{n\to\infty} \left|S^{(n)}\right|/n$.

Then consider the requirements of the definition of $E^{(n)}\left(s_1^i, \ldots, s_{q_{c_i}+1}^i, c_i, c_i^*\right)$:

1. Each college c_i ranks s_2^i last of $s_1^i, \ldots, s_{q_{c_i}+1}^i$ with independent probability $1/2$.[27]

2. Each college c_i^* ranks s_2^i higher than s_1^i with independent probability $1/2$.

3. For sufficiently high n, with probability at least $1-\epsilon/2$, at least $m \cdot \exp\left[-\overline{Q}(3k - 2\ell + 1)\right] \cdot (1-\epsilon)$ of the $c_i \in \{c_1, \ldots, c_m\}$ satisfy the condition that no students other than $s_1^i, \ldots, s_{q_{c_i}+1}^i$ find c_i or c_i^* acceptable (by Lemma A.6).

[27] This requires only $s_1 \succ_{c_1} s_2$, allowing s_2 to be the c_1-dispreferred student among $\left\{s_2, \ldots, s_{q_{c_i}+1}\right\}$ without loss of generality.

4. s_1^i finds c_i^* and c_i most desirable, in that order, with independent probability ℓ^2/nk^3.

5. s_2^i finds c_i and c_i^* most desirable, in that order, with independent probability ℓ^2/nk^3.

6. $s_3, \ldots, s_{q_{c_i}+1}$ all find c_i most desirable with independent probability $(1/n)^{\left(q_{c_i}-1\right)}$.

The probabilities of each requirement are independent, as each concerns the preferences of distinct agents. So, for sufficiently high n, with probability at least $1-\epsilon/2$, a fraction

$$\left|S^{(n)}\right| \cdot \left({}_{\left(\left|S^{(n)}\right|-1\right)}C_{q_{c_i}}\right) \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\ell^2}{nk^3} \cdot \frac{\ell^2}{nk^3} \cdot \left(\frac{1}{n}\right)^{\left(q_{c_i}-1\right)} \cdot (1-\epsilon) \tag{A.14}$$

of the colleges that satisfy condition 3 for some $s_1^i, \ldots, s_{q_{c_i}+1}^i$ also satisfy the other five conditions. For sufficiently high n,

$$\left|S^{(n)}\right| \cdot \left({}_{\left(\left|S^{(n)}\right|-1\right)}C_{q_{c_i}}\right) > \frac{\underline{Q}^{\left(q_{c_i}+1\right)} n^{1+1}}{q_{c_i}!}(1-\epsilon). \tag{A.15}$$

Thus, with probability at least $1-\epsilon$, the number of colleges in $\{c_1, \ldots, c_m\}$ that satisfy all six conditions is at least

$$m \cdot \frac{\underline{Q}^{(q+1)} \cdot \exp[-\overline{Q}(3k-2\ell+1)] \cdot \ell^4}{4k^6 \overline{q}!} \cdot (1-\epsilon)^3. \tag{A.16}$$

So, noting $\underline{Q} \leq p\overline{q}$, $\overline{Q} \geq p$, and $m \geq \frac{n}{k+1}(1-\epsilon)$ for sufficiently high n, conclude that the event $E_{c_i}^{(n)}$ occurs for at least

$$n \cdot \left(\frac{\ell^4 \min(1,p)^{(\overline{q}+1)} \exp[-p\overline{q}(3k-2\ell+1)]}{4k^6(k+1)\overline{q}!} \cdot (1-\epsilon)^4 \right) \tag{A.17}$$

of the $c_i \in C^{(n)}$ with probability bounded below by $1-\epsilon$. □

Lemma A.6. *Fix $\epsilon > 0$. There exists sufficiently large n such that, with probability at least $1-\epsilon/2$, at least $m \cdot \exp[-\overline{Q}(3k-2\ell+1)] \cdot (1-\epsilon)$ of the $c_i \in \{c_1, \ldots, c_m\}$ satisfy the condition that no $s \in S^{(n)} \backslash \left\{s_1^i, \ldots, s_{q_{c_i}+1}^i\right\}$ find c_i or c_i^* acceptable.*

Proof. Let X_i be the indicator variable of the event that c_i, c_i^* is unacceptable to all $s \in S^{(n)} \backslash \left\{ s_1^i, \ldots, s_{q_{c_i}+1}^i \right\}$. For all i, then,

$$\Pr[X_i] = \left(1 - \frac{k+1}{n} - \frac{2(k-\ell)}{n-\ell} \right)^{\left(|S^{(n)}| - q_{c_i} - 1 \right)}. \tag{A.18}$$

Thus,

$$\liminf_{n \to \infty} \mathbb{E} \left[\sum_{i=1}^{m} X_i \right] \geq m \cdot \exp\left[-\overline{Q}(3k - 2\ell + 1) \right] \tag{A.19}$$

and, for sufficiently high n,

$$\mathbb{E} \left[\sum_{i=1}^{m} X_i \right] \geq m \cdot \exp\left[-\overline{Q}(3k - 2\ell + 1) \right] \cdot (1 - \epsilon/2). \tag{A.20}$$

Note that for $i \neq j$, $\mathrm{Covar}[X_i, X_j] < 0$. Therefore, for sufficiently high n,

$$\mathrm{Var} \left[\sum_{i=1}^{m} X_i \right] < \sum_{i=1}^{m} \mathrm{Var}[X_i] < \frac{n}{k+1} \cdot \exp\left[-\overline{Q}(3k - 2\ell + 1) \right] \tag{A.21}$$

By Chebyshev's inequality, then,

$$\begin{aligned} &\Pr \left[\sum_{i=1}^{m} X_i > m \cdot \exp\left[-\overline{Q}(3k - 2\ell + 1) \right] \cdot (1 - \epsilon) \right] \\ &\geq 1 - \frac{4}{\epsilon^2 \cdot \frac{n}{k+1} \cdot \exp\left[-\overline{Q}(3k - 2\ell + 1) \right]}, \end{aligned} \tag{A.22}$$

the right-hand side of which approaches 1 as $n \to \infty$ and exceeds $1 - \epsilon/2$ for sufficiently large n. $\square$

B Figures excluded from main text

This appendix presents figures that extend § 4, reporting average core size and welfare gap size observed in market simulations for various market specifications. Plots are labeled with number of realizations observed (denoted t).

Figures 8 and 9 extend Figures 1–3 and report on one-to-one Watts–Strogatz markets of size $|C| \in \{40, 400, 1000, 4000\}$ with full locality. Core sizes and welfare gaps remain substantial (and largely constant) in unbalanced markets for small k, even as $|C|$ becomes large.

Figures 10–12 report on many-to-one Watts–Strogatz markets of $q \in \{1, 4, 10\}$ and $|C| \in \{400, 1000, 4000\}$ with full locality. While matched students' average welfare gaps shrink with increased q, gaps remain apparent to visual inspection and the fraction of colleges with incentive to manipulate remains substantial for small k.

Figure 13 extends Figure 5 and compares $k = \ell$ Watts–Strogatz markets of $|C| \in \{40, 400, 1000\}$ to markets with similarly-selective but homogeneous students. As in the $|C| = 40$ case discussed in § 4.3, the sharp welfare advantage of the short side of the market is attenuated and smoothed by selectivity in both models, though incentives to manipulate are supported only in the Watts–Strogatz model.

Figure 14 reports the fraction of unmatched agents in each experiment.

B.1 Strategic incentives in unbalanced markets

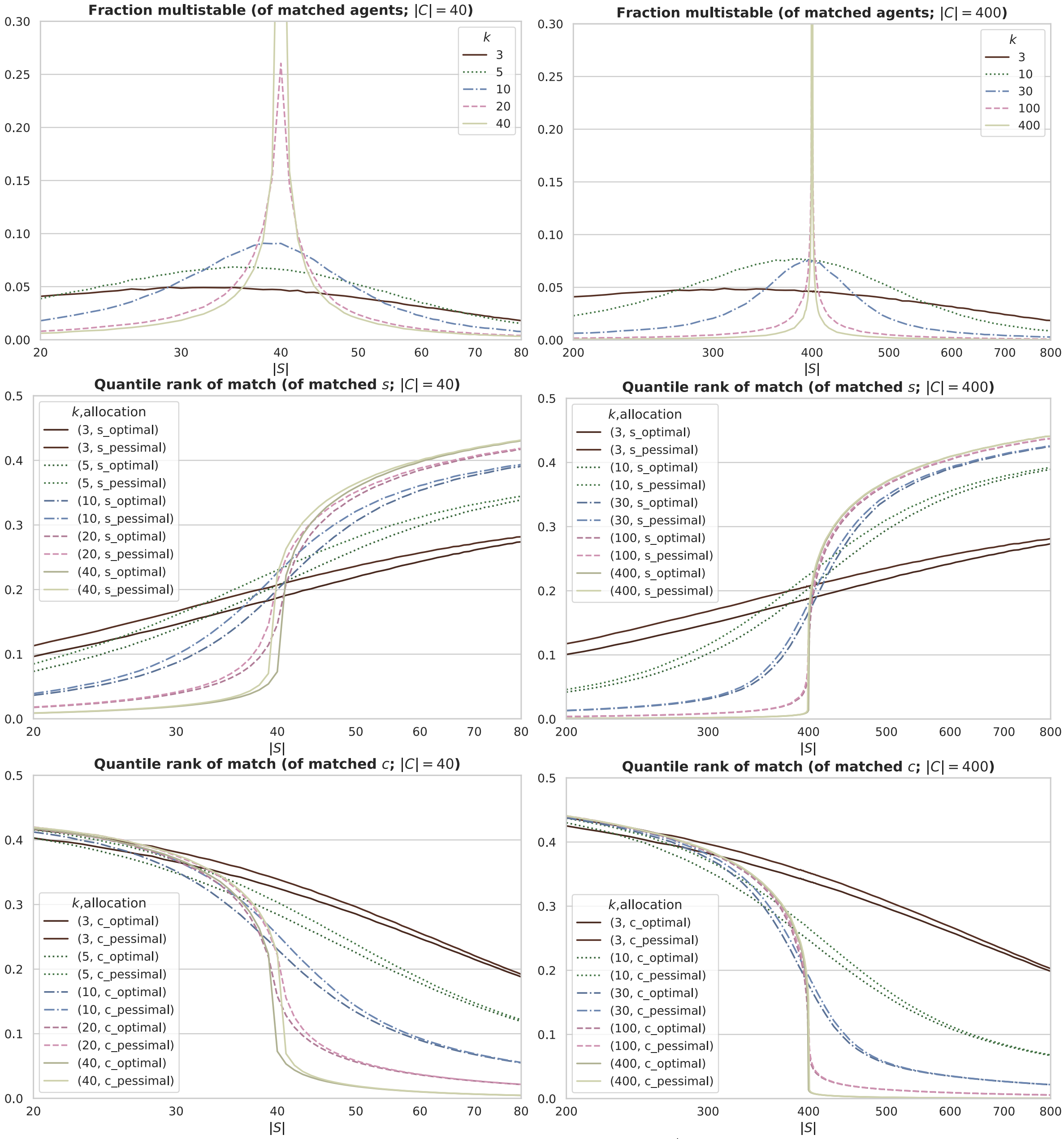

Figure 8: Average core size and matched students'/colleges' average quantile rank of matches for $|C| = 40$ (30,000 realizations) and $|C| = 400$ (3,000 realizations) under Watts–Strogatz preferences with $\ell = k$ and $q = 1$. Plots at top reproduce Figures 1, 2, and 3.

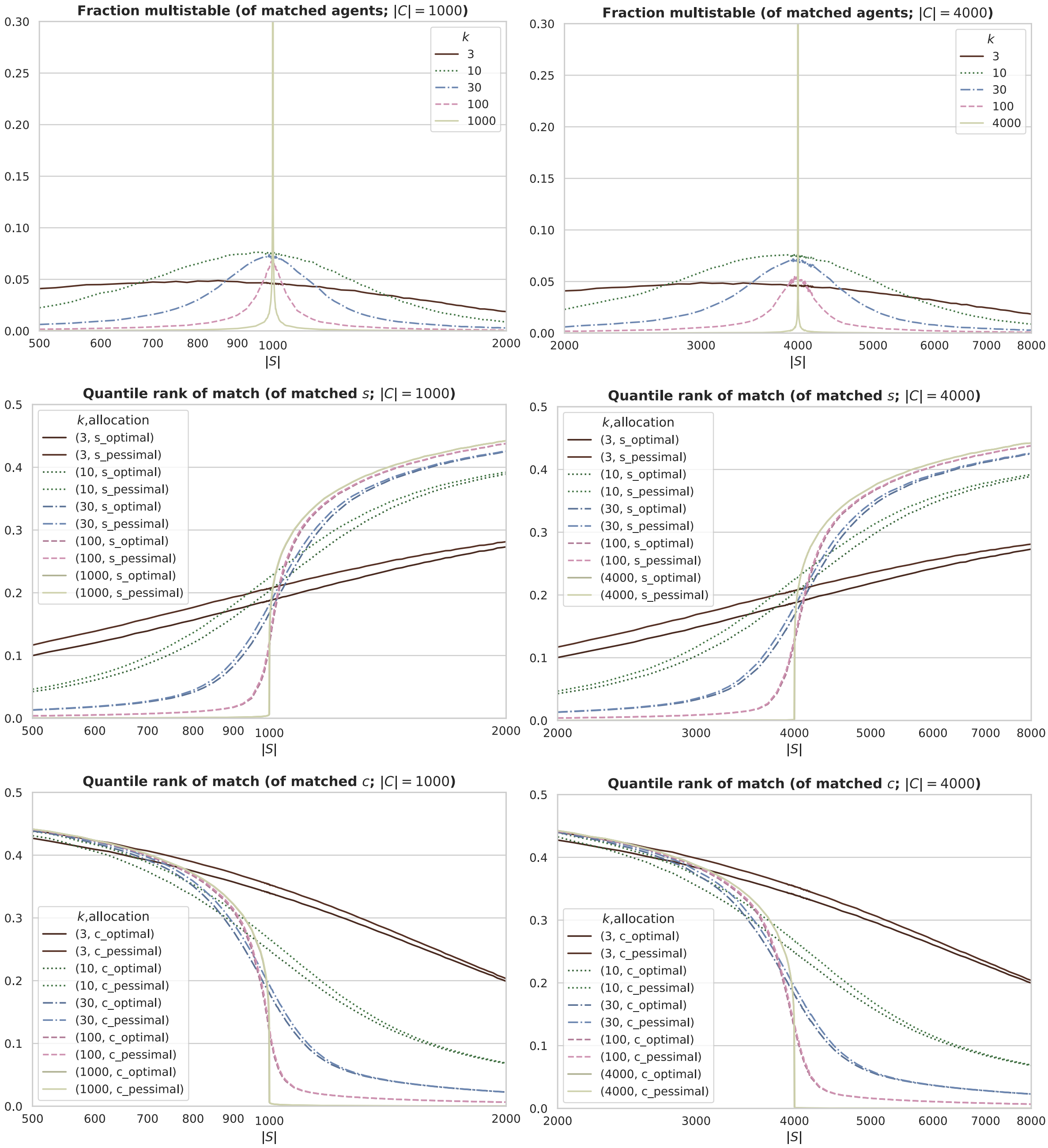

Figure 9: Average core size, matched students'/colleges' average quantile rank of matches for $|C| = 1000$ (1,000 realizations) and $|C| = 4000$ (200 realizations) under Watts–Strogatz preferences with $\ell = k$ and $q = 1$.

B.2 Core size and allocation outcomes in large markets

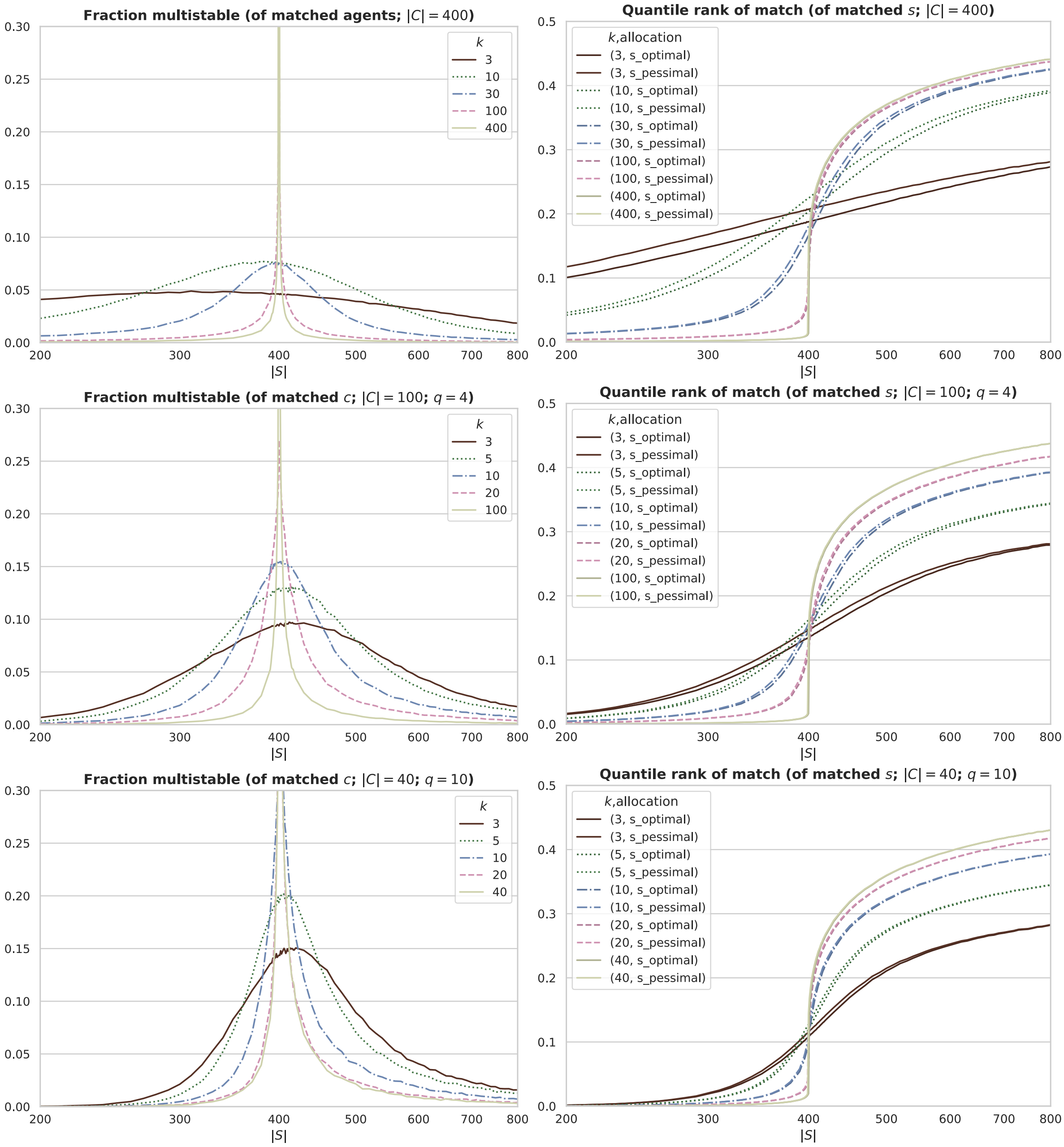

Figure 10: Average core size and matched students' average quantile rank of matches for $q \in \{1, 4, 10\}$ and $q|C| = 400$ under Watts–Strogatz preferences with $\ell = k$, across 3,000 realizations. Plots at top reproduce Figure 3.

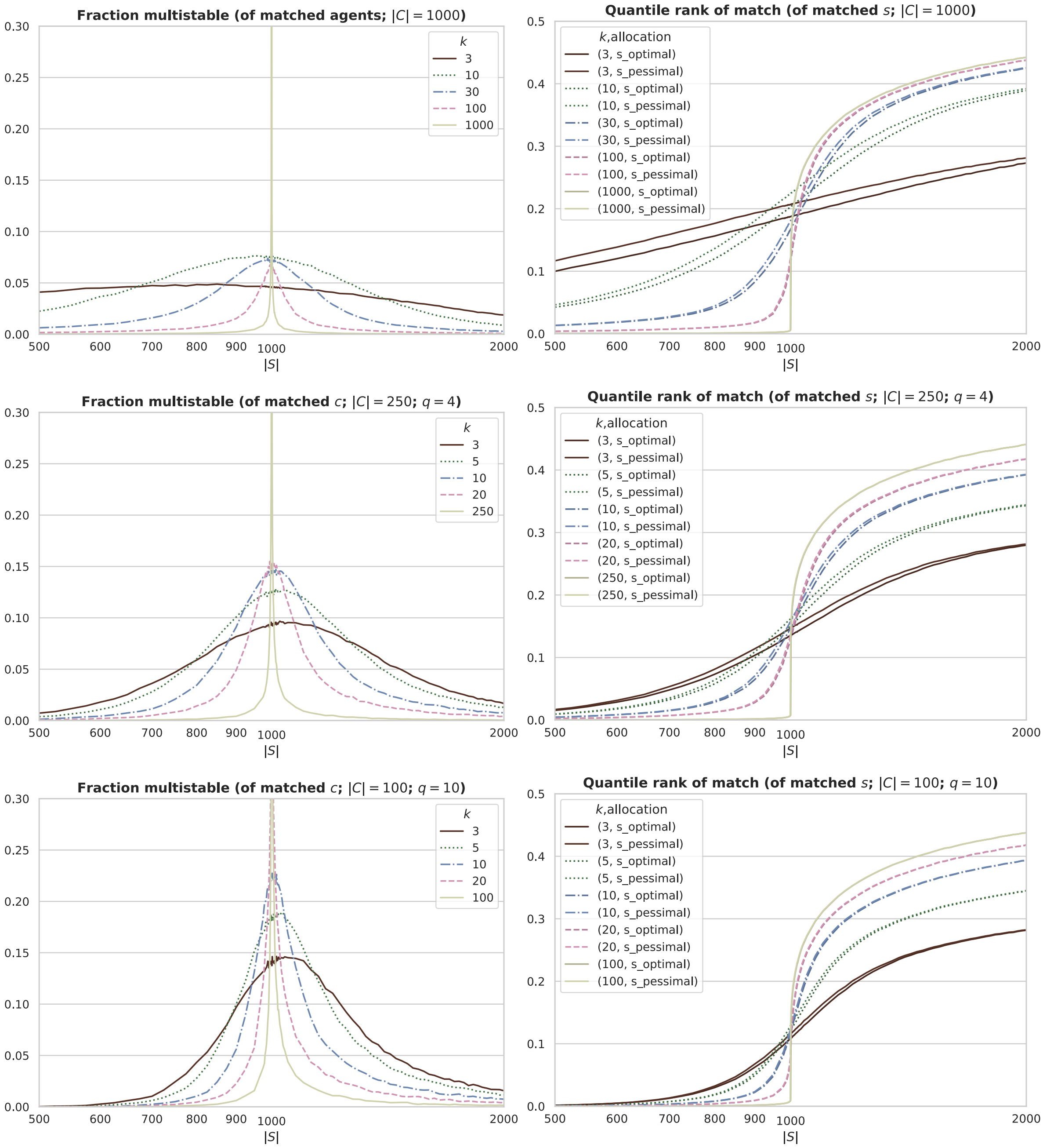

Figure 11: Average core size and matched students' average quantile rank of matches for $q \in \{1, 4, 10\}$ and $q|C| = 1000$ under Watts–Strogatz preferences with $\ell = k$, across 1,000 realizations. Plots at top reproduce parts of Figure 9.

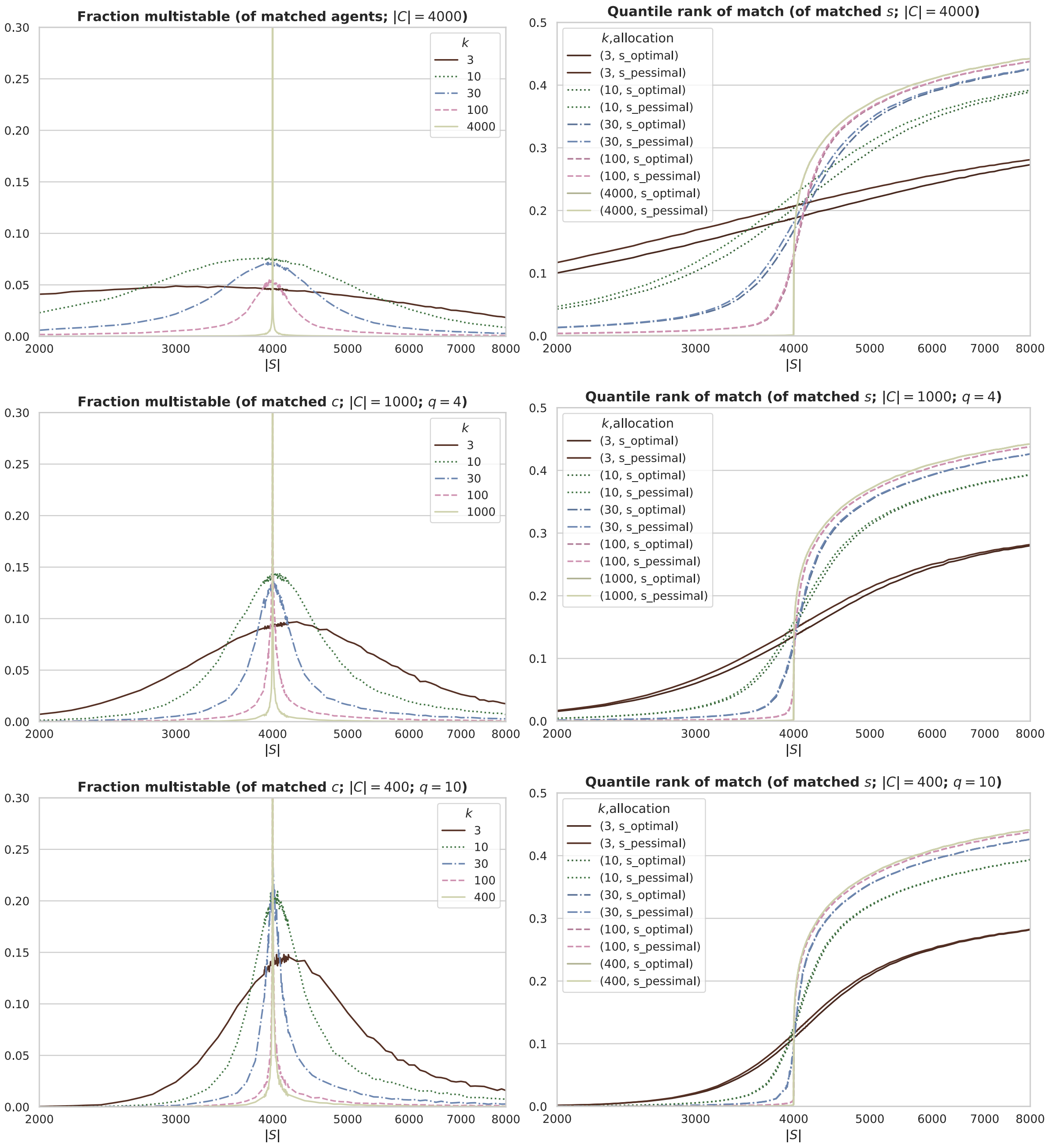

Figure 12: Average core size and matched students' average quantile rank of matches for $q \in \{1, 4, 10\}$ and $q|C| = 4000$ under Watts–Strogatz preferences with $\ell = k$, across 200 realizations. Plots at top reproduce parts of Figure 9.

B.3 Localized versus incomplete uniform preferences

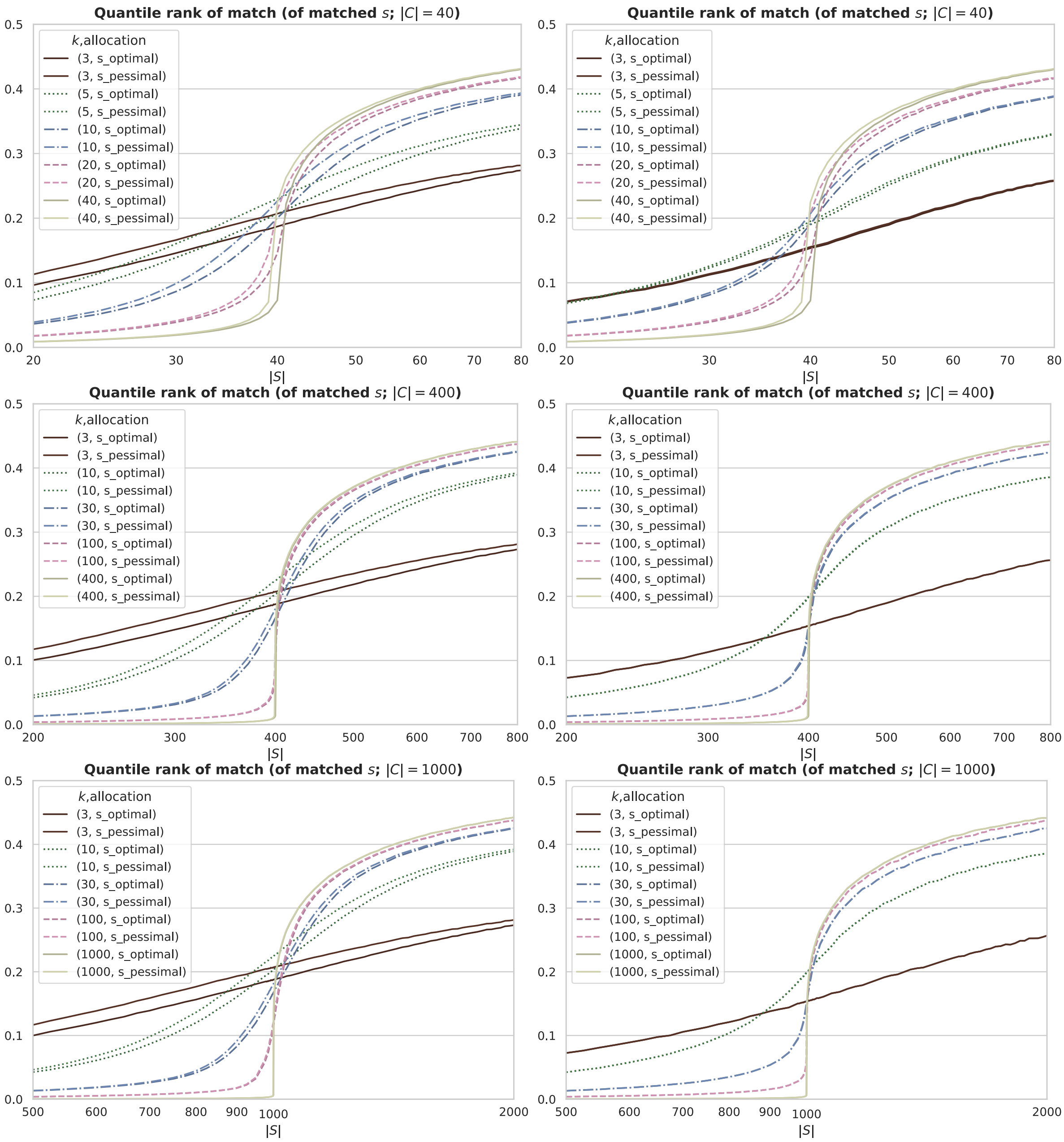

Figure 13: Matched students' average quantile rank of matches under $k = \ell$ Watts–Strogatz preferences (left) and incomplete uniform preferences (right) with $q = 1$. Plots at left reproduce parts of Figures 1, 3, and 9. Plots at right present averages across 10,000, 1,000, and 150 realizations.

B.4 Fraction of unmatched agents

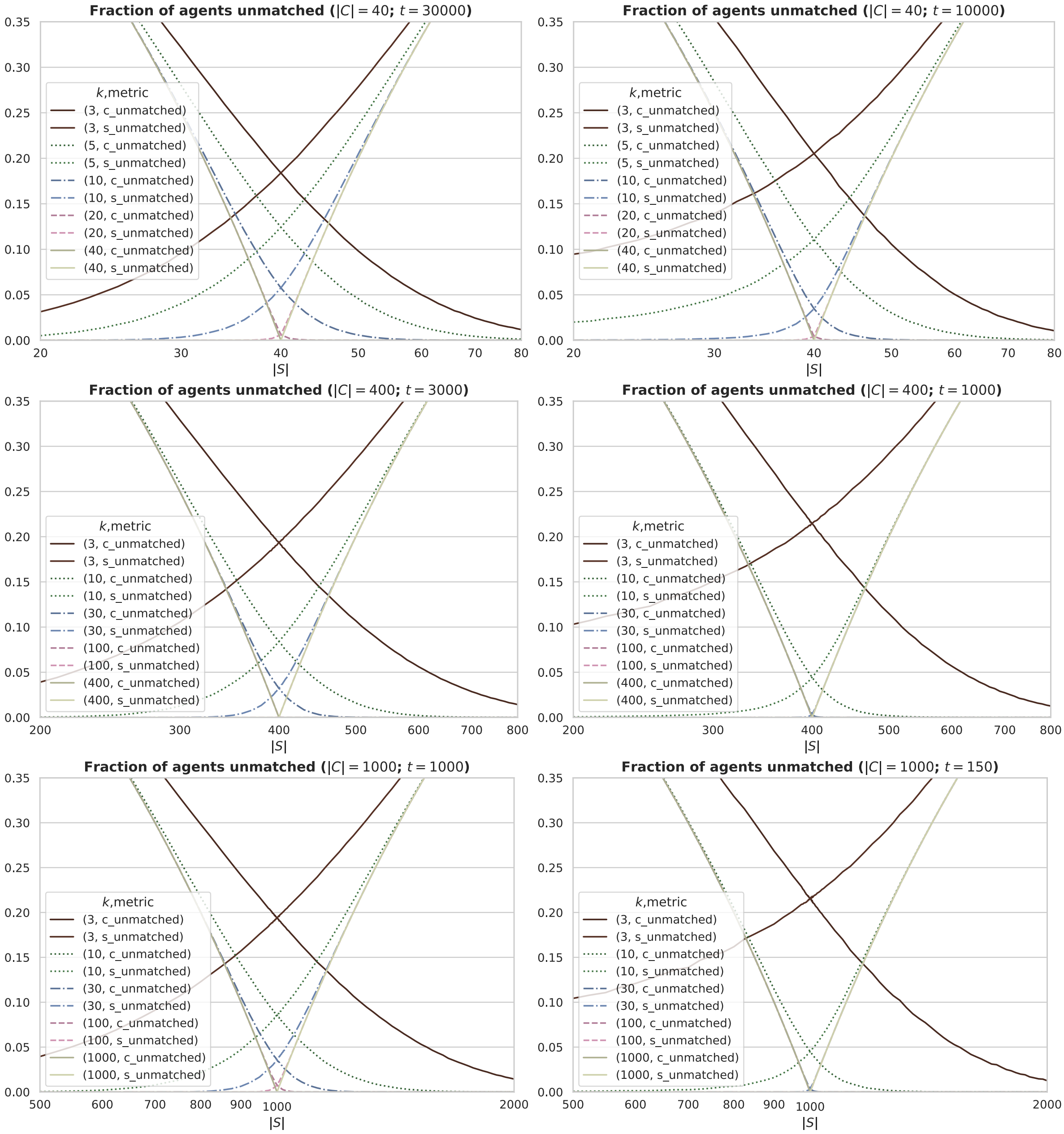

Figure 14: Average fraction of unmatched agents under $k = \ell$ Watts–Strogatz preferences with $q = 1$ (left; 30,000, 3,000, and 1,000 realizations) and incomplete uniform preferences with $q = 1$ (right; 10,000, 1,000, and 150 realizations).

C A generalized locality property

The proof of Theorem 3.1 above demonstrates the existence of non-vanishing market power in any large market model in which a variant of Lemma 3.2 obtains, namely where $\lim_{n\to\infty} E_{c_1}^{(n)} > 0$. In this appendix, I characterize a spectral condition on preference structure, with a natural interpretation as a general measure of locality, that is sufficient to demonstrate that a similar bound holds in a one-to-one market. In § C.2 below, I present many models common in the network-structure literature which satisfy the network-structural analogue of this condition. Analogous models of preference structure generate markets which exhibit non-vanishing market power.

C.1 Uniform students

Let $\tilde{\Gamma}^{(n)} = \left(C^{(n)}, S^{(n)}, \mathcal{P}_C^{(n)}, q^{(n)}, \mathcal{P}_S^{(n)}\right)$ be a random one-to-one market with $n = \left|C^{(n)}\right|$ and $\mathcal{P}_C^{(n)}$ the uniform distribution over permutations of $S^{(n)}$. Fix some c_1 and $c_2 \in C^{(n)}$, and draw a s_1 and s_2 with preferences from $\mathcal{P}_S^{(n)}$. Without loss of generality, let $s_1 \succ_{c_1} s_2$. Letting $\mathrm{r}_s(c)$ be the absolute rank of c in s's preferences, consider the following events:

- $E_{[c_1;c_2]}^{(n)} := s_2 \succ_{c_2} s_1$
- $E_{[c_1;\forall s]}^{(n)} := \forall s \in S^{(n)} \backslash \{s_1, s_2\}, (\varnothing \succ_s c_1) \wedge (\varnothing \succ_s c_2)$
- $E_{[c_1;s_1]}^{(n)} := (\mathrm{r}_{s_1}(c_2) = 1) \wedge (\mathrm{r}_{s_1}(c_1) = 2)$
- $E_{[c_1;s_2]}^{(n)} := (\mathrm{r}_{s_2}(c_1) = 1) \wedge (\mathrm{r}_{s_2}(c_2) = 2)$
- $E^{(n)}(s_1, s_2, c_1, c_2) = E_{[c_1;c_2]}^{(n)} \wedge E_{[c_1;\forall s]}^{(n)} \wedge E_{[c_1;s_1]}^{n} \wedge E_{[c_1;s_2]}^{(n)}$
- $E_{c_1}^{(n)} := \bigcup_{(s_1,s_2,c_2)\in S^{(n)}\times S^{(n)}\times C^{(n)}} E^{(n)}(s_1, s_2, c_1, c_2)$.

Since $\mathcal{P}_C^{(n)}$ is uniform over permutations of students, $\Pr\left[E_{[c_1;c_2]}^{(n)}\right] = \frac{1}{2}$. And since students' preferences are drawn independently, consider the remaining events as independent probabilities on draws of a student s from $\mathcal{P}_S^{(n)}$:

- $\Pr\left[E_{[c_1;\forall s]}^{(n)}\right] = \left(P_s[\varnothing \succ_s c_1] \cdot P_s[\varnothing \succ_s c_2 \mid \varnothing \succ_s c_1]\right)^{\left(\left|S^{(n)}\right|-2\right)}$
- $\Pr\left[E_{[c_1;s_1]}^{(n)}\right] = P_s[\mathrm{r}_s(c_2) = 1] \cdot P_s[\mathrm{r}_s(c_1) = 2 \mid \mathrm{r}_s(c_2) = 1]$

- $\Pr\left[E^{(n)}_{[c_1;s_2]}\right] = P_s[\mathrm{r}_s(c_1)=1]\cdot P_s[\mathrm{r}_s(c_2)=2 \mid \mathrm{r}_s(c_1)=1]$.

So, noting that for different selections of (s_1, s_2, c_2) the events $E^{(n)}(s_1, s_2, c_1, c_2)$ are disjoint and that there are $\frac{n(n-1)}{2}$ draws of (s_1, s_2) such that $s_1 \neq s_2$ and $s_1 \succ_{c_1} s_2$, see:

$$
\begin{aligned}
\Pr[E^{(n)}_{c_1}] =& \frac{n(n-1)}{4} \sum_{c_2 \in C^{(n)} \setminus \{c_2\}} \Pr\left[E^{(n)}_{[c_1;\forall s]}\right] \cdot \Pr\left[E^{(n)}_{[c_1;s_1]}\right] \cdot \Pr\left[E^{(n)}_{[c_1;s_2]}\right] \\
=& \frac{n(n-1)}{4} \cdot P_s[\mathrm{r}_s(c_1)=1] \cdot P_s[\varnothing \succ_s c_1]^{\left(|S^{(n)}|-2\right)} \\
&\cdot \sum_{c_2 \in C^{(n)} \setminus \{c_1\}} \Big(P_s[\mathrm{r}_s(c_2)=1] \\
&\quad \cdot P_s[\varnothing \succ_s c_2 \mid \varnothing \succ_s c_1]^{\left(|S^{(n)}|-2\right)} \\
&\quad \cdot P_s[\mathrm{r}_s(c_1)=2 \mid \mathrm{r}_s(c_2)=1] \\
&\quad \cdot P_s[\mathrm{r}_s(c_2)=2 \mid \mathrm{r}_s(c_1)=1] \Big).
\end{aligned}
\tag{C.1}
$$

We can use this expression to derive from preference structure a bound on the expected number of colleges (and students) with more than one stable match.

C.1.1 Uniform students, *ex ante* equipopular colleges

Say that colleges are *ex ante equipopular*[28] in market $\tilde{\Gamma} = (C, S, \mathcal{P}_C, q, \mathcal{P}_S)$ if

$$\forall i \in \{0, \ldots, k\}, \forall c \in C, \forall s \in S, P_s[\mathrm{r}_s(c)=i] = \frac{1}{|C|}, \tag{C.2}$$

i. e., if each college is equally likely to appear in each position in a student's preference list.

Let $\left(\tilde{\Gamma}^{(n)}\right)_{n\in\mathbb{N}}$ be a $(k, \overline{q})$-regular sequence of random one-to-one markets with colleges *ex ante* equipopular and $\mathcal{P}^{(n)}_C$ the uniform distribution over permutations of $S^{(n)}$. Consider $\tilde{\Gamma}^{(n)} = \left(C^{(n)}, S^{(n)}, \mathcal{P}^{(n)}_C, q^{(n)}, \mathcal{P}^{(n)}_S\right)$. Then

[28] Note that *ex ante* equipopularity allows for correlations within a students' rankings of different colleges, and is a weaker condition than *ex ante* symmetry (as it allows for asymmetries in correlation structure).

note that $P_s[\varnothing \succ_s c_2 \mid \varnothing \succ_s c_1] \geq \frac{n-2k}{n}$ and so:

$$\begin{aligned}\Pr[E_{c_1}^{(n)}] \geq & \frac{n(n-1)}{4} \cdot \frac{1}{n} \cdot \left(1 - \frac{k}{n}\right)^{(\overline{q}n-2)} \\ & \cdot \sum_{c_2 \in C^{(n)}} \Bigg(\frac{1}{n} \cdot \left(1 - \frac{2k}{n}\right)^{(\overline{q}n-2)} \\ & \qquad \cdot P_s[\mathrm{r}_s(c_1) = 2 \mid \mathrm{r}_s(c_2) = 1] \\ & \qquad \cdot P_s[\mathrm{r}_s(c_2) = 2 \mid \mathrm{r}_s(c_1) = 1]\Bigg),\end{aligned} \tag{C.3}$$

which in the $n \to \infty$ limit is:

$$\begin{aligned}\lim_{n\to\infty} \Pr[E_{c_1}^{(n)}] \geq \frac{\exp[-3\overline{q}k]}{4} \cdot \sum_{c_2 \in C^{(n)}} \Bigg(& P_s[\mathrm{r}_s(c_1) = 2 \mid \mathrm{r}_s(c_2) = 1] \\ & \cdot P_s[\mathrm{r}_s(c_2) = 2 \mid \mathrm{r}_s(c_1) = 1]\Bigg).\end{aligned} \tag{C.4}$$

Define the stochastic operator $A^{(n)} := \big[P_s[\mathrm{r}_s(c_i) = 2 \mid \mathrm{r}_s(c_j) = 1]\big]_{ij}$. Then note:

$$\begin{aligned}\Big(\big(A^{(n)}\big)^T\big(A^{(n)}\big)\Big)_{c_1 c_1} = \sum_{c_2 \in C^{(n)}} \Bigg(& P_s[\mathrm{r}_s(c_1) = 2 \mid \mathrm{r}_s(c_2) = 1] \\ & \cdot P_s[\mathrm{r}_s(c_2) = 2 \mid \mathrm{r}_s(c_1) = 1]\Bigg)\end{aligned} \tag{C.5}$$

and

$$\lim_{n\to\infty} \mathbb{E}\big[c \in C^{(n)} : E_c^{(n)}\big] \geq \frac{\exp[-3\overline{q}k]}{4} \lim_{n\to\infty} \mathrm{Tr}\Big(\big(A^{(n)}\big)^T\big(A^{(n)}\big)\Big). \tag{C.6}$$

That is, we can express a sufficient condition that a market exhibits non-vanishing market power (under the above regularity and uniformity assumptions) as a spectral condition the matrix of first- and second-choice colleges by students.

As an example, we can demonstrate non-vanishing market power in the setup of Theorem 3.1[29] by noting *ex ante* equipopularity, regularity, uniformity of colleges' preferences, and that $\mathrm{Tr}\big(\big(A^{(n)}\big)^T\big(A^{(n)}\big)\big) = \frac{n}{k}$, implying that the expected fraction of colleges with more than one stable match is non-vanishing.

[29] albeit with a weaker bound in terms of k and $\overline{q}$

C.2 Locality in network structure

Modeling locality in inter-agent networks was an early motivation for random graph models that generalized uniform (Gilbert (1959); Erdős and Rényi (1960)) and homogeneous power-law (Barabási and Albert (1999); Albert and Barabási (2000)) network models. For the results in the main text, I considered an analogue of a synthetic locality model presented by Watts and Strogatz (1998), which mixed an Erdős–Rényi uniform graph model with a regular ring lattice. In the network-structure literature, locality was also introduced to generative models via variation to their sequential-attachment mechanisms (Kleinberg *et al.* (1999); Kumar *et al.* (2000); Leskovec *et al.* (2005)). Further work has refined the modeling of locality structure in both synthetic (Leskovec *et al.* (2010)) and ad-hoc (Yang and Leskovec (2014)) network models.

The locality condition established above is a form of subgraph density condition on the ordered, directed graph of agent preferences, similar to the triangle-counting interpretation of network locality in undirected graphs (Wasserman and Faust (1994)). As such, a number of proposed models from the network-structure literature have preference-structure analogues that exhibit non-vanishing locality, in the above sense:

- lattice mixture models (Watts and Strogatz (1998)), as introduced in the main text,
- geography-based models (Jin *et al.* (2000)),
- copying-based (Kleinberg *et al.* (1999); Kumar *et al.* (2000)) and prototype-copying-based (Leskovec *et al.* (2005)) preferential attachment models,
- Kronecker-product models (Chakrabarti *et al.* (2004); Leskovec *et al.* (2010)),
- community-attribute models (Yang and Leskovec (2014)).

I anticipate scope for future work to investigate the applicability of network models to modeling preference structure in matching markets—and the matching-market properties that they influence.