Large random matching markets with localized preference structures can exhibit large cores

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Full paper: `https://static.rossry.net/papers/ps-core_wine20.pdf`

*These slides present independent work and only the opinions of the named author.

The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design

By ALVIN E. ROTH AND ELLIOTT PERANSON*

We report on the design of the new clearinghouse adopted by the National Resident Matching Program, which annually fills approximately 20,000 jobs for new physicians. Because the market has complementarities between applicants and between positions, the theory of simple matching markets does not apply directly. However, computational experiments show the theory provides good approximations. Furthermore, the set of stable matchings, and the opportunities for strategic manipulation, are surprisingly small. A new kind of "core convergence" result explains

(*AER*, 1999)

Collective dynamics of 'small-world' networks

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Networks of coupled dynamical systems have been used to model biological oscillators[1–4], Josephson junction arrays[5,6], excitable media[7], neural networks[8–10], spatial games[11], genetic control networks[12] and many other self-organizing systems. Ordinarily, the connection topology is assumed to be either completely regular or completely random. But many biological, technological and social networks lie somewhere between these two extremes. Here we explore simple models of networks that can be tuned through this middle ground: regular networks 'rewired' to intro-

(*Nature*, 1998)

Design considerations for matching markets:

- Stability is crucial to matching market design.
- Size of the core is closely related to strategic incentives.

Received wisdom in matching markets:

- Core size is generically small.
- Relative core size shrinks in the limit as market size grows.
- Preference correlation increases "core collapse".

When agent preferences exhibit locality:

- Cores remain robustly large.
- Relative core size fails to shrink in the large-market limit.
- The effects of "correlation" are sensitive to correlation *structure*.

Presenting:

- **Background** on related prior results.
- **Model of preference locality** that robustly supports large cores.
- **Results of simulations** comparing non-/localized preferences.
- **Discussion of alternative explanations** of observed small cores.

Strategy-proofness in matching markets

Matching markets:

- L-agents (students) seek *matches* to R-agents (colleges).
- Solution concept: ex-post *stability*.
- Desired feature: ex-ante incentive-compatibility (*strategy-proofness*).

Strategy-proofness:

- Impossible in the general case.
  - Multiple stable matchings $\{\mu_1, \mu_2, \ldots\}$ may exist.
  - Agent x dishonestly rejects $\mu_1(x)$
    $\implies$ any stable-matching-finding mechanism returns μ_2 instead.
- A stable mechanism M is strategy-proof <u>for agent x</u>
  $\iff$ M always returns the stable matching that x most prefers.
- A stable mechanism can be strategy-proof for (nearly) all agents
  $\iff$ (nearly) only one stable match exists.

Core size (1/3)

Simple case:

- n students and n colleges form complete preferences unif. at random $\implies$ most agents have many stable matches.

Roth and Peranson (1999):

- Empirically, few stable matchings in the NRMP.
- All students agree (completely) on which colleges are better $\implies$ few (one) stable matchings.
- Students rank limited options $\implies$ few stable matchings.

"The theorems explaining why the core must converge as it does will surely follow." (Roth and Peranson (1999))

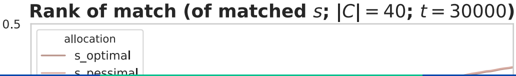
- One-to-one setting: Immorlica and Mahdian (2005).
- Many-to-one setting: Kojima and Pathak (2009).

Core size (23/3)Matching with contracts?

- n students, n colleges, uniform choices $\Longrightarrow$ **large** core.
- Students agree on college ordering $\Longrightarrow$ **small** core. (RP '99)
- Students rank limited options $\Longrightarrow$ **small** core. (IM '05, KP '09)
- Cross-side imbalance $\Longrightarrow$ **small** core. (AKL '15)

Ashlagi, Kanoria, and Leshno (2015):

- Cross-side imbalance (#students $\neq$ #colleges).
- Dramatic welfare implications:
  - "[E]ach agent on the long side is either unmatched or does almost no better than being matched to a random partner."
- Also $\Longrightarrow$ **small** core.
- Robust to certain models of preference correlation.



Matching markets

Defining a *matching market* $\Gamma := (C, S, (\succ_c)_C, (\succ_s)_S, (q_c)_C)$:

- Students S with preferences $\succ_s$ over $C \cup \{\varnothing\}$.
- Colleges C with preferences $\succ_c$ over 2^S, *responsive with quota* q_c.
  - Responsive pref. $\iff$ additive utility func. (Kojima and Pathak (2009))
- Matching $\mu : C \cup S \to C \cup \{\varnothing\} \cup 2^S$ satisfying:
  - $\mu(c) \in 2^S$.
  - $\mu(s) \in C \cup \{\varnothing\}$.
  - $\mu(s) = c \iff s \in \mu(c)$.

Defining *stable* (= core) matchings:

- Matching μ *blocked* by (s, c) iff:
  - $c \succ_s \mu(s)$.
  - $\exists s' \in \mu(c)$: either $s \succ_c s'$ or $|\mu(c)| < q_c$ and $s \succ_c \varnothing$.
- A matching is *stable* (and in the core) iff unblocked by any (s, c).

N.B.: As standard, measure *core size* in fraction of students (colleges) with multiple stable matches. (Immorlica and Mahdian (2005))

Random matching markets

Define a *random market* $\tilde{\Gamma} := (C, S, \mathcal{P}_C, \mathcal{P}_S, (q_c)_C)$

- Each college c draws $\succ_c$ from $\mathcal{P}_C$.
- Each student s draws $\succ_s$ from $\mathcal{P}_S$.

$(k, \bar{q})$-*regular* sequence of random markets (growing in #agents):

- $\left(\tilde{\Gamma}^{(1)}, \tilde{\Gamma}^{(2)}, \ldots\right)$, with $\tilde{\Gamma}^{(1)} = \left(C^{(1)}, S^{(1)}, \mathcal{P}_C^{(1)}, \mathcal{P}_S^{(1)}, \left(q_c^{(1)}\right)_{C^{(1)}}\right)$.
- $\left|C^{(n)}\right| = n$, $\left|S^{(n)}\right| \leq \bar{q}n$. (regularized #students)
- $\forall n$, all $\tilde{\succsim}_s$ supported in $\mathcal{P}_S^{(n)}$ have length k. (fixed-length pref. lists)
- $\forall n, \forall c \in C^{(n)}, q_c^{(n)} \leq \bar{q}$. (bounded college quotas)

(p, r)-*unbalanced*: $\left|S^{(n)}\right| = r + p \sum_{c \in C} q_c^{(n)}$.

Preferences with homogeneous structure

Immorlica & Mahdian ('05), Kojima & Pathak ('09), Ashlagi *& al.* ('15):

- Fix preference list size k.
- Fix $\mathcal{D} = (p_c)_C$ a probability distribution on C.
- $\mathcal{P}_S \sim$ draw k times from $\mathcal{D}$ without replacement.

$\mathcal{P}_C \sim$ uniform, with all students acceptable to all colleges.

Results (one-to-one and many-to-one settings):

- In a $(k, \bar{q})$-regular sequence of balanced random markets, the fraction of students (and colleges) with > 1 stable match vanishes as $n \to \infty$.
- Likewise, the fraction of agents who can manipulate DA vanishes.

Ashlagi, Kanoria, Leshno (2015):

- Letting $k = n$ (students rank *all* colleges), a $\bar{q}$-regular sequence of (p, r)-unbalanced markets has vanishing core (as $n \to \infty$) if $p \neq 1$.

Preferences with locality

Simple *geographic locality* for $\mathcal{P}_S$:

- Fix preference list size k.
- Arrange the colleges C uniformly on a circle.
- Students draw a location on the circle (uniformly at random), then draw preferences uniformly from the k nearest colleges.

$\mathcal{P}_C \sim$ uniform, with all students acceptable to all colleges.

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cf. regular ring lattice networks (Watts and Strogatz, 1998); compare homogeneous case to (weighted) Erdős–Rényi networks.

Main theorems

Theorem (3.3, Core size)

Given a $(k, \bar{q})$-regular sequence of (p, r)-unbalanced random markets with k-nearest-colleges geographic student preferences, there exists $\Delta > 0$ s.t.:

- *Δ bounds below the fraction of $C^{(n)}$ with > 1 stable match.*
- *Δ bounds below the fraction of $S^{(n)}$ with > 1 stable match.*

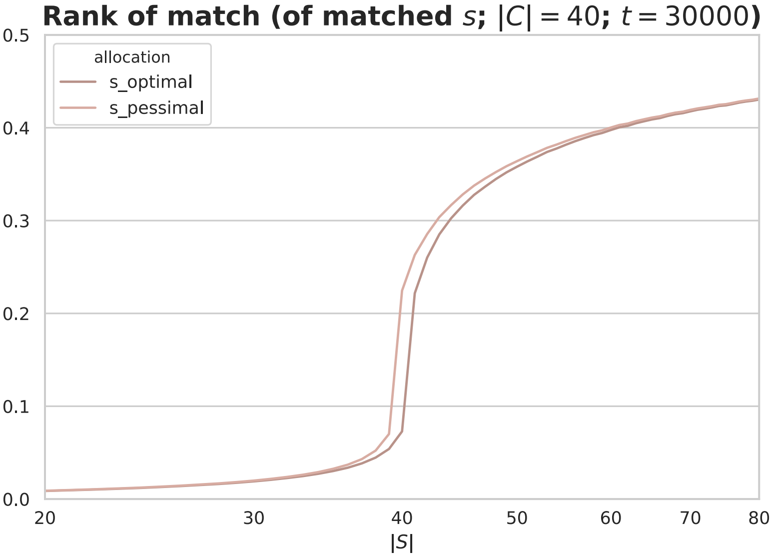
Core size fails to vanish in the $n \to \infty$ limit.

Corollary (3.4, Strategy-proofness)

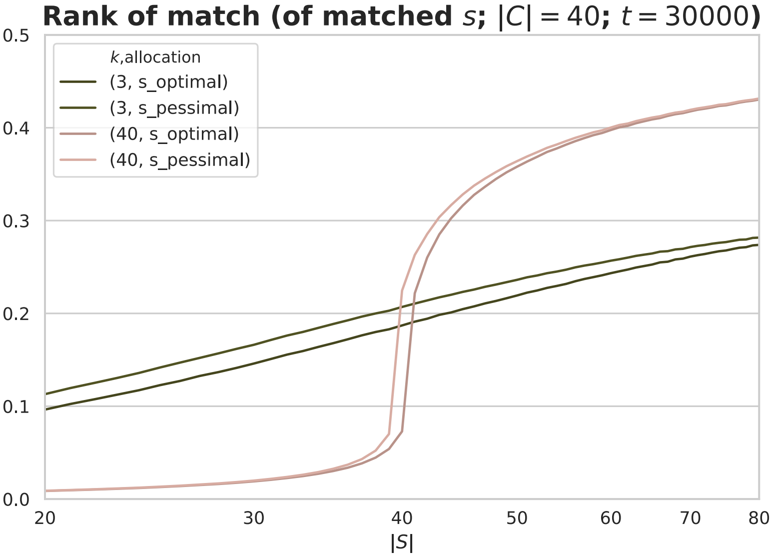
Given a $(k, \bar{q})$-regular sequence of (p, r)-unbalanced random markets with k-nearest-colleges geographic student preferences, there exists $\Delta > 0$ s.t.:

- *Δ bounds below the fraction of agents who can manipulate DA.*

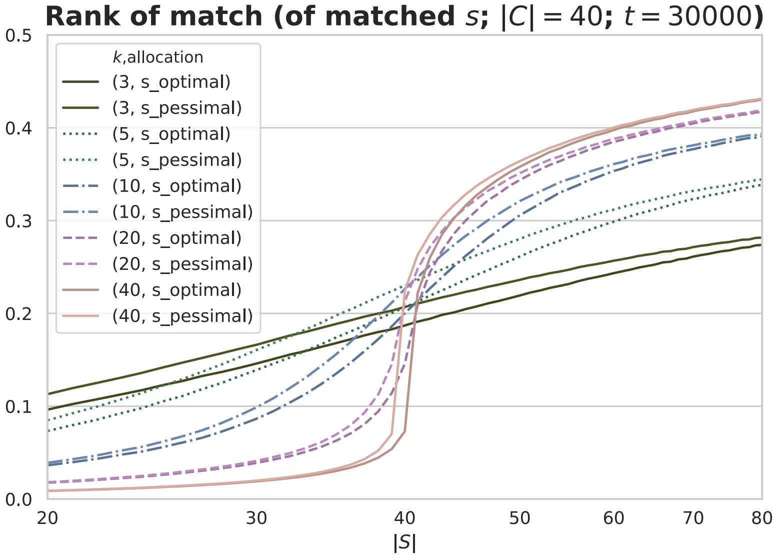
Rank-of-match under uniform preferences



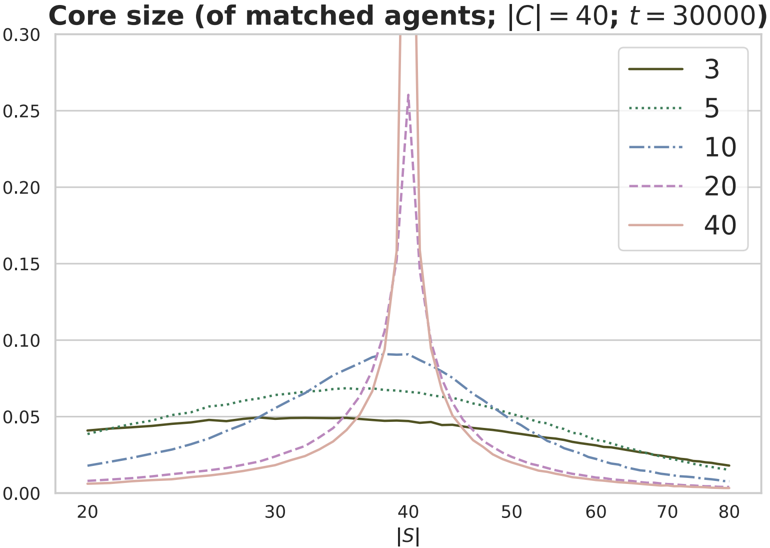
Rank-of-match under preference locality



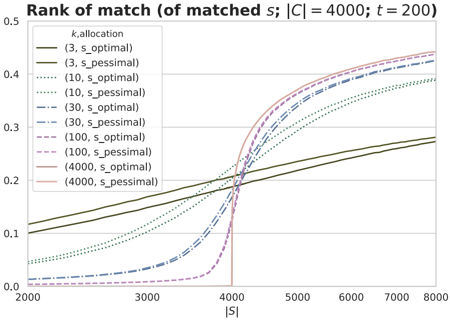
Rank-of-match under preference locality

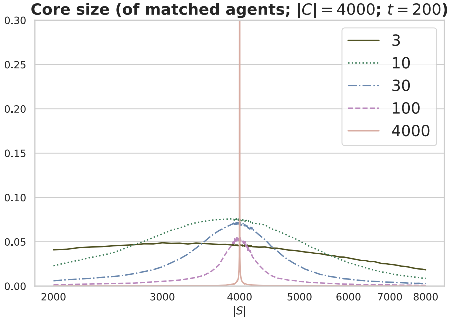


Core size under preference locality



Match dynamics under preference locality in large markets





Main result (proof, one-to-one balanced)

$(k, 1)$-reg. seq., balanced random mkt., k-nearest-colleges preferences:

- $\Delta > 0$ bounds below the fraction of $C^{(n)}$ with > 1 stable match.

- $s_1 \succ_{c_1} s_2 \succ_{c_1} \ldots$ (among students feasible for c_1)
- $s_2 \succ_{c_2} s_1 \succ_{c_2} \ldots$ (among students feasible for c_2)
- $c_2 \succ_{s_1} c_1 \succ_{s_1} \ldots$ (among all colleges) $\longrightarrow$ $\boxed{p(\bullet) = \frac{1}{nk}}$
- $c_1 \succ_{s_2} c_2 \succ_{s_2} \ldots$ (among all colleges) $\longrightarrow$ $\boxed{p(\bullet) = \frac{1}{nk}}$

Probability of all conditions (for fixed c_1, s_1, s_2, and $c_2 := c_1 + 1$):

$$\liminf_{n\to\infty} p(c_1; s_1; s_2) \geq \liminf_{n\to\infty} \frac{1}{2} \cdot \left(1 - \frac{k+1}{n-1}\right)^{(n-2)} \cdot \frac{1}{nk} \cdot \frac{1}{nk} = \frac{\exp[-k-1]}{\boxed{2n^2k^2}}$$

$$n(n-1)/2 \text{ choices of } s_1, s_2 \Longrightarrow \liminf_{n\to\infty} p(c_1) \geq \frac{\exp[-k-1]}{\boxed{4k^2}} \cdot \gamma =: \Delta$$

Main result (proof, one-to-one unbalanced)

$(k, 1)$-reg. seq., (p, r)-unbalanced rand. mkt., k-nearest-colleges pref.:

- $\Delta > 0$ bounds below the fraction of $C^{(n)}$ with > 1 stable match.

- $s_1 \succ_{c_1} s_2 \succ_{c_1} \dots$ (among students feasible for c_1)
- $s_2 \succ_{c_2} s_1 \succ_{c_2} \dots$ (among students feasible for c_2)
- $c_2 \succ_{s_1} c_1 \succ_{s_1} \dots$ (among all colleges) $\longrightarrow p(\bullet) = \frac{1}{nk}$
- $c_1 \succ_{s_2} c_2 \succ_{s_2} \dots$ (among all colleges) $\longrightarrow p(\bullet) = \frac{1}{nk}$

Probability of all conditions (for fixed c_1, s_1, s_2, and $c_2 := c_1 + 1$):

$$\liminf_{n\to\infty} p(c_1; s_1; s_2) \geq \frac{\boxed{\min\left(1, p^2\right)} \exp[-\boxed{p}(k+1)]}{2n^2k^2}$$

$$\text{choice of } s_1, s_2 \Longrightarrow \liminf_{n\to\infty} p(c_1) \geq \frac{\boxed{\min\left(1, p^2\right)} \exp[-\boxed{p}(k+1)]}{4k^2} \cdot \gamma$$

Summary

- **Discussion of previously-studied factors** affecting core size.
- **Model of preference locality** that robustly supports large cores.
- **Results of simulations** comparing non-/localized preferences.

- **Discussion of alternative explanations** of observed small cores.

Explanations for small cores

- Large, unbalanced matching markets need not have small cores.
- Empirically, many markets exhibit small cores anyway.

- Very strong global preference alignment.
- Preference formation/expression as a binding constraint.
  - Search costs (a role for platforms?), or
  - "Simultaneous search" (Shorrer 2019).

...would imply true cores (much?) larger than observed cores.

- Localized matching dynamics.
  - Localized cross-side imbalance.
  - Strong preference alignment at local scales.
- "Pre-manipulated" submitted preferences.

\end{talk}